MICROECONOMICS
Final exam – October 19th, 2012
Eric Danan

3 hours – Documents and calculators NOT allowed

Exercise 1^{B}

Assume $L \geq 2$, let $\alpha, \gamma \in \mathbb{R}^L_+$ with $\alpha \gg 0$, and consider the consumption set

$$X = \{x \in \mathbb{R}^L_+ : x \geq \gamma\}$$

and a competitive consumer whose utility function $u : X \to \mathbb{R}$ is defined by

$$u(x) = \prod_{l=1}^{L} (x_l - \gamma_l)^{\alpha_l}.$$

(This is known as a Stone-Geary utility function.) Throughout this exercise we restrict attention to price-wealth situations (p, w) such that $p \gg 0$ and $w > p \cdot \gamma$. We can interpret γ as a "subsistence" bundle that has to be consumed in order to merely "survive", the consumer only drawing utility from the "residual bundle" $x - \gamma$.

1. To what kind of preferences over residual bundles does u correspond?

> **Solution:** To Cobb-Douglas preferences over residual bundles.

2. Why can you assume without loss of generality that $\sum_{l=1}^{L} \alpha_l = 1$? Do so from now on.

> **Solution:** Because this corresponds the increasing transformation $u \mapsto u^{\frac{1}{\sum_{l=1}^{L} \alpha_l}}$.

3. Show that the consumer's utility maximization program always has a unique solution and that this solution is always interior.

> **Solution:** Since u is continuous, the utility maximization program always has a solution. Since u is strictly concave (as a product of positive strictly concave functions), the solution is always unique. Since $u(x) > 0$ for any interior x and $u(x) = 0$ for any non-interior x, the solution is always interior.

4. Determine the consumer's Walrasian demand function. Interpret. What is the shape of the wealth expansion path?

Solution: Using the first-order conditions and the budget constraint, we obtain

$$x_l(p,w) = \gamma_l + \frac{\alpha_l}{p_l}(w - p \cdot \gamma) \text{ for } l = 1, \ldots, L.$$

The interpretation is that the consumer first buys the subsistence bundle γ, which costs him $p \cdot \gamma$, and then chooses the best residual bundle $x - \gamma$ he can afford with his residual wealth $w - p \cdot \gamma$. Since he has Cobb-Douglas preferences over the residual bundles, the latter step is achieved by allocating a fixed share α_l of residual wealth to each good l.
The wealth expansion path is a straight line with strictly positive slope.

5. Determine the consumer's indirect utility function.

Solution: Plugging the Walrasian demand function into the utility function, we obtain

$$v(p,w) = (w - p \cdot \gamma) \prod_{l=1}^{L} \left(\frac{\alpha_l}{p_l}\right)^{\alpha_l}.$$

6. Without solving the consumer's expenditure minimization program, determine his expenditure and Hicksian demand functions.

Solution: Using the indirect utility function and solving $v(p, e(p,u)) = u$, we obtain:

$$e(p,u) = p \cdot \gamma + u \prod_{l=1}^{L} \left(\frac{p_l}{\alpha_l}\right)^{\alpha_l}.$$

Hence, by Shephard's lemma:

$$h_l(p,u) = \gamma_l + \frac{u\alpha_l}{p_l} \prod_{k=1}^{L} \left(\frac{p_k}{\alpha_k}\right)^{\alpha_k} \text{ for } l = 1, \ldots, L.$$

7. Compute the own-price and cross-price partial derivatives of the Walrasian and Hicksian demand functions. Interpret their signs.

Solution: For the own-price partial derivatives, we obtain

$$\begin{aligned}\frac{\partial x_l(p,w)}{\partial p_l} &= -\frac{\alpha_l\gamma_l}{p_l} - \frac{\alpha_l}{p_l^2}(w - p\cdot\gamma) < 0 \\ \frac{\partial h_l(p,u)}{\partial p_l} &= -\frac{\alpha_l(1-\alpha_l)}{p_l^2}u\prod_{k=1}^{L}\left(\frac{p_k}{\alpha_k}\right)^{\alpha_k} < 0\end{aligned} \qquad \text{for } l = 1,\ldots,L$$

That the latter is negative is expected from the compensated law of demand. That the former is negative means that no good is a Giffen good at any price-wealth situation, which is also expected from Cobb-Douglas preferences (for which all goods are normal and no good is Giffen) over residual bundles (the first negative term is a wealth effect corresponding to the variation in residual wealth, and the second negative term combines a wealth and a substitution effects at constant residual wealth). For the cross-price partial derivatives, we obtain

$$\begin{aligned}\frac{\partial x_l(p,w)}{\partial p_k} &= -\frac{\alpha_l\gamma_k}{p_k} < 0 \\ \frac{\partial h_l(p,u)}{\partial p_k} &= \frac{\alpha_l\alpha_k}{p_l p_k}u\prod_{m=1}^{L}\left(\frac{p_m}{\alpha_m}\right)^{\alpha_m} > 0\end{aligned} \qquad \text{for } l,k = 1,\ldots,L, l\neq k$$

That the latter is negative means that all goods are substitutes to one another, which is expected from Cobb-Douglas preferences over residual bundles. That the former is positive means that all goods are gross complements to one another, which indicates the presence of a wealth effect (given that Cobb-Douglas Walrasian demand functions have zero cross-price derivatives, the positive sign corresponds to the variation in residual wealth).

8. Verify that the Slutsky equation holds.

Solution: Given that $\partial x_l(p,w)/\partial w = \alpha_l/p_l$ for $l = 1,\ldots,L$, we can check that

$$\frac{\partial h_l(p,v(p,w))}{\partial p_l} = -\frac{\alpha_l(1-\alpha_l)}{p_l^2}(w - p\cdot\gamma) = \frac{\partial x_l(p,w)}{\partial p_l} + \frac{\partial x_l(p,w)}{\partial w}x_l(p,w)$$

for $l = 1,\ldots,L$ and that

$$\frac{\partial h_l(p,v(p,w))}{\partial p_k} = \frac{\alpha_l\alpha_k)}{p_l p_k}(w - p\cdot\gamma) = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w}x_k(p,w)$$

for $l,k = 1,\ldots,L$, $l\neq k$.

Exercise 2^B

Consider a competitive producer whose single-output, two-input technology is defined by the production function

$$q = f(z) = \max(z_l^{\frac{1}{4}}(z_k - 1)^{\frac{1}{4}}, 0),$$

where $z = (z_l, z_k) \geq 0$ denotes the vector of input quantities (l for labor and k for capital) and $q \geq 0$ denotes the output quantity. Let $w = (w_l, w_k) \gg 0$ denote the vector of input prices and $p > 0$ denote the output price.

1. What properties does the production set satisfy? How are the returns to scale?

> **Solution:** The production set is non-empty, closed and satisfies free disposal. There is setup cost of 1 unit of capital, which is not sunk since there is possibility of inaction ($f(0) = 0$). f is strictly concave for $z_k \geq 1$ but is not globally concave because of the setup cost, so the production set is not convex. Hence the returns to scale are neither decreasing nor increasing nor constant (they would be strictly decreasing without the setup cost).

2. Determine the conditional factor demand correspondences (or functions) $z_l(w, q)$ and $z_k(w, q)$ and the cost function $C(w, q)$.

> **Solution:** Obviously, we have $z_l(w, 0) = z_k(w, 0) = C(w, 0) = 0$. For $q > 0$ we must have $z_l > 0$ and $z_k > 1$, so using the production function $q = z_l^{\frac{1}{4}}(z_k - 1)^{\frac{1}{4}}$ and the first-order condition of the cost minimization program $MRTS_{lk} = \frac{w_l}{w_k}$ we obtain
>
> $$z_l(w, q) = w_l^{-\frac{1}{2}} w_k^{\frac{1}{2}} q^2,$$
>
> $$z_k(w, q) = \begin{cases} w_l^{\frac{1}{2}} w_k^{-\frac{1}{2}} q^2 + 1 & \text{if } q > 0, \\ 0 & \text{if } q = 0. \end{cases}$$
>
> The cost function is then given by
>
> $$C(w, q) = w_l z_l(w, q) + w_k z_k(w, q) = \begin{cases} w_k + 2w_l^{\frac{1}{2}} w_k^{\frac{1}{2}} q^2 & \text{if } q > 0, \\ 0 & \text{if } q = 0. \end{cases}$$

3. Determine the marginal cost function $MC(w, q)$, the average cost function $AC(w, q)$, the efficient scale $\bar{q}(w)$, the minimum price $\bar{p}(w)$ enabling the producer to make a non-negative profit, and the supply correspondence (or function) $q(w, p)$. Represent them graphically.

Solution: The marginal and average costs are then given (for $q > 0$) by

$$\begin{aligned} MC(w,q) &= \frac{\partial C(w,q)}{\partial q} = 4w_l^{\frac{1}{2}}w_k^{\frac{1}{2}}q, \\ AC(w,q) &= \frac{C(w,q)}{q} = \frac{w_k}{q} + 2w_l^{\frac{1}{2}}w_k^{\frac{1}{2}}q. \end{aligned}$$

Solving either $\frac{\partial AC(w,q)}{\partial q} = 0$ or $AC(w,q) = MC(w,q)$, we obtain

$$\begin{aligned} \bar{q}(w) &= 2^{-\frac{1}{2}}w_l^{-\frac{1}{4}}w_k^{\frac{1}{4}}, \\ \bar{p}(w) &= AC(w,\bar{q}(w)) = 2^{\frac{3}{2}}w_l^{\frac{1}{4}}w_k^{\frac{3}{4}}. \end{aligned}$$

Solving the first-order condition of the profit maximization program $MC(w,q) = p$ for $p \geq p^*(w)$, we then obtain the supply correspondence

$$q(w,p) = \begin{cases} \{0\} & \text{if } p < \bar{p}(w), \\ \left\{0, 2^{-2}w_l^{-\frac{1}{2}}w_k^{-\frac{1}{2}}p\right\} & \text{if } p = \bar{p}(w), \\ \left\{2^{-2}w_l^{-\frac{1}{2}}w_k^{-\frac{1}{2}}p\right\} & \text{if } p > \bar{p}(w). \end{cases}$$

The graphical illustration is Figure 5.D.4(c), with a linear marginal cost.

4. Determine the profit function $\pi(w,p)$ and the (unconditional) factor demand correspondences (or functions) $\hat{z}_l(w,p)$ and $\hat{z}_k(w,p)$.

Solution: Using the supply correspondence $q(w,p)$, we obtain

$$\pi(w,p) = pq(w,p) - C(w,q(w,p)) = \begin{cases} 0 & \text{if } p \leq \bar{p}(w), \\ 2^{-3}w_l^{-\frac{1}{2}}w_k^{-\frac{1}{2}}p^2 - w_k & \text{if } p > \bar{p}(w), \end{cases}$$

$$\hat{z}_l(w,p) = z_l(w,q(w,p)) = \begin{cases} \{0\} & \text{if } p < \bar{p}(w), \\ \left\{0, 2^{-4}w_l^{-\frac{3}{2}}w_k^{-\frac{1}{2}}p^2\right\} & \text{if } p = \bar{p}(w), \\ \left\{2^{-4}w_l^{-\frac{3}{2}}w_k^{-\frac{1}{2}}p^2\right\} & \text{if } p > \bar{p}(w), \end{cases}$$

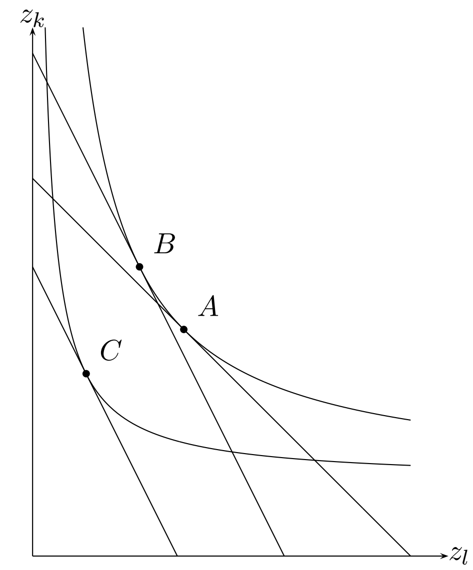
$$\hat{z}_k(w,p) = z_k(w,q(w,p)) = \begin{cases} \{0\} & \text{if } p < \bar{p}(w), \\ \left\{0, 2^{-4}w_l^{-\frac{1}{2}}w_k^{-\frac{3}{2}}p^2 + 1\right\} & \text{if } p = \bar{p}(w), \\ \left\{2^{-4}w_l^{-\frac{1}{2}}w_k^{-\frac{3}{2}}p^2 + 1\right\} & \text{if } p > \bar{p}(w). \end{cases}$$

5. Restricting attention to price situations (w, p) for which the supply correspondence is single-valued and strictly positive, compute the effect of a marginal increase $dw_l > 0$ of the price of labor on the conditional and unconditional factor demands as a function of (w, p). Determine their respective signs and compare, for each input, the conditional and the unconditional effect. Interpret and illustrate graphically in the input space.

Solution: We restrict attention to price situations (w, p) such that $p > \bar{p}(w)$. The effects are then simply given by the partial derivatives of the conditional and unconditional factor demands with respect to w_l (evaluating the conditional ones at $q = q(w, p)$):

$$
\begin{aligned}
dz_l &= \frac{\partial z_l(w, q(w,p))}{\partial w_l} dw_l = -2^{-3} w_l^{-\frac{5}{2}} w_k^{-\frac{1}{2}} p^2 dw_l, \\
dz_k &= \frac{\partial z_k(w, q(w,p))}{\partial w_l} dw_l = 2^{-3} w_l^{-\frac{3}{2}} w_k^{-\frac{3}{2}} p^2 dw_l, \\
d\hat{z}_l &= \frac{\partial \hat{z}_l(w,p)}{\partial w_l} dw_l = -3 \cdot 2^{-3} w_l^{-\frac{5}{2}} w_k^{-\frac{1}{2}} p^2 dw_l, \\
d\hat{z}_k &= \frac{\partial \hat{z}_k(w,p)}{\partial w_l} dw_l = -2^{-3} w_l^{-\frac{3}{2}} w_k^{-\frac{3}{2}} p^2 dw_l.
\end{aligned}
$$

Hence $d\hat{z}_l < dz_l < 0$ and $d\hat{z}_k < 0 < dz_k$.
The conditional effects correspond to a substitution of labor for capital at a given output level: as labor becomes more costly relative to capital, the cost-minimizing input bundle to produce a given output level contains less labor and more capital (graphically, as the iso-profit lines become steeper, the tangency point on a given iso-quant moves from A to B). The unconditional effects also take into account the effect of the price change on the output level: as some inputs becomes more costly, the profit-maximizing output level decreases, which in turn reduces the cost-minimizing quantities of all inputs (graphically, as the iso-quant moves towards the origin, the tangency point moves from B to C). This is why the unconditional effect are always lower than the conditional ones. For labor both effects go in the same direction of reducing demand, and hence so does the total (i.e. unconditional) effect. For capital the two effect go in opposite directions, but the latter overweighs the former, so the total effect is a reduction of demand as well.



Exercise 3[A]

Assume $L \geq 2$ and consider a competitive consumer whose Walrasian demand function $x(p, w)$ is defined by

$$x_l(p, w) = \frac{\sum_{k=1}^{L} \beta_{lk} p_k}{\sum_{k=1}^{L} p_k} \cdot \frac{w}{p_l} \text{ for } l = 1, \ldots, L,$$

where all β_{lk}'s are non-negative parameters.

1. For which values of the parameters is this demand function homogeneous of degree 0?

> **Solution:** It is easy to check that it is homogeneous of degree zero for all values of the parameters, see Exercise 2.E.1.

2. For which values of the parameters does this demand function satisfy Walras' law?

> **Solution:** Walras' law is satisfied if and only if $\sum_{l=1}^{L} \sum_{k=1}^{L} \beta_{lk} p_k = \sum_{l=1}^{L} p_l$ for all p. The left-hand side of this equality is equal to $\sum_{l=1}^{L} (\sum_{k=1}^{L} \beta_{kl}) p_l$. Hence Walras' law is satisfied if and only if $\sum_{k=1}^{L} \beta_{kl} = 1$ for all l.

Exercise 4[A]

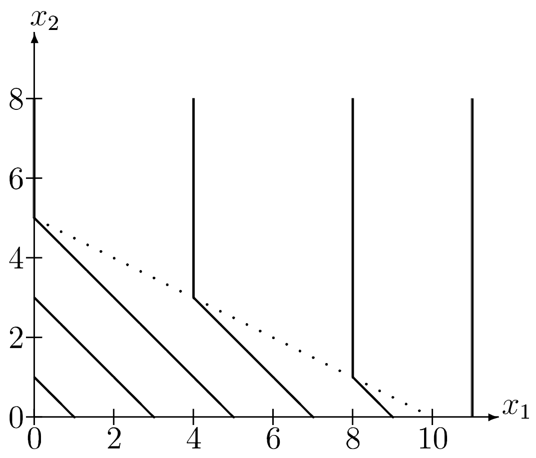
(This exercise is inspired by P N Sorrensen (2007), *Economic Theory* 31:367–370.) Consider a competitive consumer whose utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ is defined by

$$u(x) = \min(x_1 + 10, 2(x_1 + x_2)).$$

Answer all the questions of this exercise graphically.

1. Draw (a representative subset of) the consumer's indifference curves.

Solution:

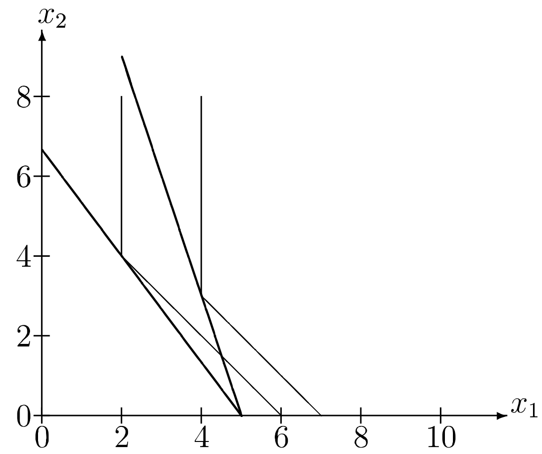


2. Are the consumer's preferences monotone? strongly monotone? convex? continuous? homothetic?

Solution: They are clearly monotone, not strongly monotone, convex, continuous, and not homothetic.

3. Does the consumer's Walrasian demand satisfy the uncompensated law of demand? the compensated law of demand?

Solution: It does not satisfy the uncompensated law of demand because good 2 is a Giffen good at some price-wealth situations:



It does satisfy the compensated law of demand because it is generated by a preference relation.

Exercise 5[A]

Consider a preference relation $\succsim$ on a set $X = \{x_n : n \in N\}$ of alternatives, where the index set N is a nonempty (finite or infinite) set of positive integers. The goal of this exercise is to show that $\succsim$ is complete and transitive if and only if there exists a utility function $u : X \to \mathbb{R}$ representing $\succsim$. We start by establishing the "only if" part.

1. Show that if there exists a utility function $u : X \to \mathbb{R}$ representing $\succsim$ then $\succsim$ is complete and transitive.

Solution: See the proof of Proposition 1.B.2 (the proposition holds for an arbitrary set X of alternatives).

We now turn to the "if" part, so assume $\succsim$ is complete and transitive. For all $n \in N$, let $L(n) = \{m \in N : x_n \succsim x_m\}$ denote the (index set of the) lower contour set of x_n and let $|L(n)|$ denote the cardinality (i.e. the number of elements) of $L(n)$.

2. Show that if N is finite then the $u(x_n) = |L(n)|$ is a utility function representing $\succsim$.

Solution: u can be shown to represent $\succsim$ as in Exercise 1.B.5.

3. Why is u no longer a utility function representing $\succsim$ if X is infinite?

Solution: If N is infinite then $u(x_n) = +\infty$ for some $n \in N$, so u is not a real-valued function.

4. (Harder) Find a utility function $\hat{u}$ that represents $\succsim$ whether N is finite or infinite.

Solution: Let (for instance) $\hat{u}(x) = \sum_{m \in L(n)} \frac{1}{2^m}$. Intuitively, in $u(x_n)$ each $x_m \in L(n)$ counts for 1 whereas in $\hat{u}(x_n)$ each $x_m \in L(n)$ only counts for $\frac{1}{2^m}$. This makes $\hat{u}(x_n)$ finite (it ranges between 0 and 1) for all $n \in N$, whether N is finite or infinite, so $\hat{u}$ is a real-valued function. Since each $x_m \in L(n)$ still counts positively in $\hat{u}(x_n)$, $\hat{u}$ can be shown to represent $\succsim$ in exactly the same way as u is shown to represent $\succsim$ in the finite case.