Microeconomics – solutions to problem set 2

Eric Danan*

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Problem 1 *(Binary Relations, Preferences, Indifference Curves)*

a) (*i*) BS and WS are open; (*ii*) BS and WS are open; (*iii*) BS and WS are closed; (*iv*) WS are open, BS are closed; (*v*) BS are open, WS are closed.

b) Continuity is defined on lecture slide 21. Only (*iii*) is continuous. ((*iv*) is upper-hemicontinuous, (*v*) is lower-hemicontinuous).

Problem 2 *(Preferences, quasi-concavity)*

a) If $\succsim$ is convex, then for any x and $y \in X$, and any $\alpha \in (0;1)$, $x \succsim z$ and $y \succsim z$ imply $\alpha x + (1-\alpha) y \succsim z$. In particular, if $x \succsim y$, then $\alpha x + (1-\alpha) y \succsim y$. Since u represents the preference, we have that if $u(x) \geq u(y)$, then

$$u(\alpha x + (1-\alpha) y) \geq u(y) = \min\{u(x); u(y)\}$$

or, $u(\cdot)$ is quasi-concave.

b) If u is quasi-concave, then the better sets of u for a given z, i.e., the sets $\{y \mid u(y) \geq u(z)\}$ are convex. It follows that if $u(x) \geq u(z)$ and $u(y) \geq u(z)$, then

$$u(\alpha x + (1-\alpha) y) \geq u(z)$$

Since $u(\cdot)$ represents $\succsim$, we obtain that $x \succsim z$ and $y \succsim z$ imply $\alpha x + (1-\alpha) y \succsim z$, or the convexity of $\succsim$. .

c) Note that the preference relation $\succsim$ represented by $u(\cdot)$ has a "thick" indifference curve given by the set $\{x \mid x_1 + x_2 \in [b; 2b]\}$. Suppose that $v(\cdot)$ is a utility function representing the same preferences as u. Take x, y and z such that $z = \alpha x + (1-\alpha) y$

*THEMA, University of Cergy-Pontoise, CNRS. E-mail: eric.danan@u-cergy.fr. Webpage: www.ericdanan.net.

and such that $x_1+x_2 \in (b; 2b)$, $z_1+z_2 \in (b; 2b)$ and $y_1+y_2 > 2b$. Note that according to $\succsim$ represented by $u(\cdot)$, $y \succ x \sim z$. Hence, $v(y) > v(x) = v(z)$. Now note that

$$v(\alpha x + (1-\alpha) y) = v(z) = v(x) < \alpha v(x) + (1-\alpha) v(y)$$

and hence, v violates concavity.

Problem 3 *(Preferences, monotonicity and non-satiation)*

a) From strict monotonicity, we have that for any x, $y \in X$ such that $y \geq x$, $y \neq x$, $y \succ x$. Since $y >> x$ implies $y \geq x$ and $y \neq x$, we have $y \succ x$. Hence, monotonicity is satisfied.

b) From monotonicity, for any x, $y \in X$ such that $y >> x$, $y \succ x$. Note that for any x, and any $\epsilon > 0$, there is a y such that $|x - y| < \epsilon$ and $y >> x$. By monotonicity, $y \succ x$, and non-satiation is satisfied.

c) Suppose that $y >> x$. By weak monotonicity, $y \succsim x$. Furthermore, there exists an ϵ such that $0 < \epsilon < \min\{y_1 - x_1 ... y_n - x_n\}$ and for any such ϵ, if $||z - x|| < \epsilon$, then $z << y$ and, by weak monotonicity, $y \succsim z$. Finally, by non-satiation, there is a z such that $||z - x|| < \epsilon$ and $z \succ x$. Since $y \succsim z$ and $z \succ x$, transitivity implies $y \succ x$.

Problem 4 *(Homothetic and quasi-linear preferences)*

a) Suppose that $\succsim$ can be represented by a function u which is homogeneous of degree 1. Then, $u(x) = u(y)$ is satisfied iff $u(\alpha x) = u(\alpha y)$, or, $x \sim y$ is equivalent to $ax \sim ay$ and the preference is homothetic. Conversely, suppose that the preference $\succsim$ is homothetic. Since the preference is complete, transitive and continuous, there exists a continuous utility function u which represents $\succsim$. W.l.o.g., let $u(x) \geq 0$ (apply a monotone transformation if necessary) for all $x \in X$. Since $x \sim y$ is equivalent to $\alpha x \sim \alpha y$ for all α, we must have that $u(x) = u(y)$ is satisfied iff $u(\alpha x) = u(\alpha y)$. This property is equivalent to u being a homogenous function. Suppose that u is homogenous of degree $r > 0$ and define v as

$$v(x) = [u(x)]^{\frac{1}{r}}$$

for all $x \in X$. Since v is a monotone transformation of u and since v is homogenous of degree 1, we obtain a representation of $\succsim$ by a function homogenous of degree 1.

b) If $\succsim$ can be represented by $u(x) = x_1 + v(x_2 ... x_L)$, then $u(x) = u(y)$ is satisfied if and only if $u(x + (\alpha; 0...0)) = u(y + (\alpha; 0...0))$ for all α. Hence, $x \sim y$ iff $x +$

$(\alpha;0...0) \sim y+(\alpha;0...0)$ for all α and the preference is quasi-linear w.r.t. commodity 1. Suppose now that $\succsim$ is quasi-linear w.r.t. good 1. Since it is complete, transitive and continuous, there is a continuous utility function u which represents $\succsim$ and hence, by quasi-linearity satisfies $u(x) = u(y)$ if and only if $u(x+(\alpha;0...0)) = u(y+(\alpha;0...0))$ It follows that for arbitrary x and y,

$$u(x+(\alpha;0...0)) - u(y+(\alpha;0...0)) = u(x) - u(y)$$

$$u(x+(\alpha;0...0)) - u(x) = u(y+(\alpha;0...0)) - u(y) =: v(\alpha)$$

Hence, for an arbitrary consumption bundle, x, u satisfies

$$u(x) = u(0;x_2...x_L) + v(x_1)$$

where v is a strictly increasing function in x_1, as long as utility is strictly increasing in commodity 1. Now take two arbitrary values x_1 and y_1 and let[1] $(x_2...x_L)$, $(y_2...y_L)$ be such that $x \sim y$. Since

$$u(x) = u(0;x_2...x_L) + v(x_1) = u(y) = u(0;y_2...y_L) + v(y_1)$$

and, by quasi-linearity, for any α,

$$u(x+(\alpha;0...0)) = u(0;x_2...x_L)+v(x_1+\alpha) = u(y+(\alpha;0...0)) = u(0;y_2...y_L)+v(y_1+\alpha)$$

we conclude that

$$v(x_1+\alpha) - v(x_1) = v(y_1+\alpha) - v(y_1) = v(\alpha)$$

for all x_1 and y_1, (where the last equality follows from the fact that we can choose $y_1 = 0$). Hence, for any x_1 and α_1,

$$v(x_1+\alpha) = v(\alpha) + v(x_1)$$

and v satisfies additivity. Now consider $v(kx_1)$ If k is natural, the additivity property implies that $v(kx_1) = kv(x_1)$ and therefore, setting $y_1 = kx_1$, $v\left(\frac{y_1}{k}\right) = \frac{1}{k}v(y_1)$. If $k = \frac{p}{m}$ is rational, then $v\left(\frac{p}{m}x_1\right) = pv\left(\frac{x_1}{m}\right) = \frac{p}{m}v(x_1)$ Finally, since v is continuous, and since every real number k can be approximated by a sequence of rational

[1] I am taking a short-cut here — for some x_1 and y_1 the corresponding $(x_2...x_L)$, $(y_2...y_L)$ might not exist. Then, you need to look at intervals $[x_1;y_1]$ for which the corresponding $(x_2...x_L)$, $(y_2...y_L)$ exist and piece them together to get the function v. This makes for a much longer proof — see the solution manual to MasCollel, Whinston and Green for details. The main intuition, however is clear from what follows.

number $k_1 ... k_n ...$, we have that

$$v(kx_1) = \lim_{n\to\infty} v(k_n x_1) = \lim_{n\to\infty} k_n v(x_1) = kv(x_1)$$

Hence, v also satisfies scaling. A function that satisfies additivity and scaling is a linear function. Hence, a quasi-linear continuous preference $\succsim$ can be represented by a quasi-linear function.