Microeconomics – solutions to problem set 3

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Problem 1 *(Utility Maximization, Kuhn-Tucker Method, Demand Theory, Quasi-Linear Utility)*

a) Chris' utility maximization problem:

$$\max_{\left\{(x_1 \ldots x_L) \in \mathbb{R}_+^L\right\}} \left\{ x_1 + \sum_{l=2}^{L} (x_l)^\beta \mid \sum_{l=1}^{L} x_l p_l \leq y \right\}$$

$$\mathcal{L}(x_1; x_2; \lambda_1; \lambda_2; \lambda_3) = x_1 + x_2^\beta + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 (y - p_1 x_1 - p_2 x_2)$$

The first-order conditions are:

$$\begin{aligned} (i)\ \frac{\partial \mathcal{L}(x_1; x_2; \lambda_1; \lambda_2; \lambda_3)}{\partial x_1} &= 1 + \lambda_1 - \lambda_3 p_1 = 0 \\ (ii)\ \frac{\partial \mathcal{L}(x_1; x_2; \lambda_1; \lambda_2; \lambda_3)}{\partial x_2} &= \beta x_2^{\beta-1} + \lambda_2 - \lambda_3 p_2 = 0 \end{aligned}$$

$$\begin{aligned} (iii)\ \lambda_1 x_1 &= 0 \\ (iv)\ \lambda_2 x_2 &= 0 \\ (v)\ \lambda_3 (y - p_1 x_1 - p_2 x_2) &= 0 \end{aligned}$$

$$\begin{aligned} (vi)\ x_1 &\geq 0, x_2 \geq 0 \\ (vii)\ y &\geq p_1 x_1 + p_2 x_2 \\ (viii)\ \lambda_1, \lambda_2, \lambda_3 &\geq 0 \end{aligned}$$

If $\beta = 1$, the solution to this problem is given on the lecture slides.

For $\beta \neq 1$, note that if $\lambda_3 = 0$, we have $1 + \lambda_1 = 0$, or $\lambda_1 = -1$, in contradiction to $\lambda_1 \geq 0$. I.e., since the utility function is monotone, the budget constraint will always

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hold with equality. It follows that we can consider only those cases, in which $\lambda_3 > 0$ and $y = p_1 x_1 + p_2 x_2$.

Let $\beta < 1$ and suppose that $x_2 = 0$, and hence, $x_1 = \frac{y}{p_1} > 0$. It follows that $\lambda_1 = 0$. Since $\beta < 1$, $\lim_{x_2 \to 0} \frac{\beta}{x_2^{1-\beta}} = \infty$, hence there are no λ_2 and λ_3 which would satisfy condition (ii).

If $x_1 = 0$, then $x_2 = \frac{y}{p_2} > 0$. It follows that $\lambda_2 = 0$, $\lambda_3 = \frac{\beta}{y^{1-\beta} p_2^\beta}$

$$\lambda_1 = \lambda_3 p_1 - 1 = \frac{\beta p_1}{y^{1-\beta} p_2^\beta} - 1 \geq 0$$

if $\frac{y^{1-\beta} p_2^\beta}{\beta p_1} \leq 1$. Hence, for $\frac{\beta p_1}{p_2^\beta} \geq y^{1-\beta}$, $x_1 = 0$, $x_2 = \frac{y}{p_2}$, $\lambda_1 = \frac{\beta p_1}{y^{1-\beta} p_2^\beta} - 1$, $\lambda_2 = 0$ and $\lambda_3 = \frac{\beta}{y^{1-\beta} p_2^\beta}$ define a solution for conditions (i) – $(viii)$.

Finally, if $x_1 > 0$ and $x_2 > 0$, we have $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \frac{1}{p_1}$, $x_2 = \left(\frac{\beta p_1}{p_2}\right)^{\frac{1}{1-\beta}}$, $x_1 = \frac{y - \left(\frac{\beta p_1}{p_2}\right)^{\frac{1}{1-\beta}} p_2}{p_1} \geq 0$, or

$$\begin{aligned} y - \left(\frac{\beta p_1}{p_2}\right)^{\frac{1}{1-\beta}} p_2 &\geq 0 \\ y^{1-\beta} &\geq \frac{\beta p_1}{p_2^\beta} \end{aligned}$$

Hence, for $y^{1-\beta} \geq \frac{\beta p_1}{p_2^\beta}$, $x_1 = \frac{y - \left(\frac{\beta p_1}{p_2}\right)^{\frac{1}{1-\beta}} p_2}{p_1}$, $x_2 = \left(\frac{\beta p_1}{p_2}\right)^{\frac{1}{1-\beta}}$, $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \frac{1}{p_1}$ define a solution for conditions (i) – $(viii)$. Note that when $\beta < 1$, the function $x_1 + x_2^\beta$ is concave (and hence, quasi-concave). Hence, the Kuhn-Tucker conditions are necessary and sufficient for the existence of an optimum. To summarize, Chris' demand is given by:

$$f_1(p_1; p_2; y) = \begin{cases} \frac{y - \left(\frac{\beta p_1}{p_2}\right)^{\frac{1}{1-\beta}} p_2}{p_1}, & y^{1-\beta} \geq \frac{\beta p_1}{p_2^\beta} \\ 0, & y^{1-\beta} < \frac{\beta p_1}{p_2^\beta} \end{cases}$$

$$f_2(p_1; p_2; y) = \begin{cases} \left(\frac{\beta p_1}{p_2}\right)^{\frac{1}{1-\beta}}, & y^{1-\beta} \geq \frac{\beta p_1}{p_2^\beta} \\ \frac{y}{p_2}, & y^{1-\beta} < \frac{\beta p_1}{p_2^\beta} \end{cases}$$

Let $\beta > 1$ and suppose that $x_2 = 0$ and, hence, $x_1 = \frac{y}{p_1} > 0$. It follows that $\lambda_1 = 0$, $\lambda_2 = \lambda_3 p_2$ and $\lambda_3 p_1 = 1$. Hence, $x_1^* = \frac{y}{p_1}$, $x_2^* = 0$ together with $\lambda_1 = 0$, $\lambda_3 = \frac{1}{p_1}$, $\lambda_2 = \frac{p_2}{p_1}$ define a solution for conditions (i) – $(viii)$.

Next, suppose that $x_1 = 0$ and hence, $x_2 = \frac{y}{p_2} > 0$. It follows that $\lambda_2 = 0$, $\lambda_3 = \frac{\beta y^{\beta-1}}{p_2^\beta}$,

$$\lambda_1 = \lambda_3 p_1 - 1 = \frac{\beta y^{\beta-1} p_1}{p_2^\beta} - 1 \geq 0$$

if $\frac{\beta y^{\beta-1} p_1}{p_2^\beta} \geq 1$. Hence, for $y^{\beta-1} \geq \frac{p_2^\beta}{\beta p_1}$, $x_1 = 0$, $x_2 = \frac{y}{p_2}$, $\lambda_1 = \frac{\beta y^{\beta-1} p_1}{p_2^\beta} - 1$, $\lambda_2 = 0$ and $\lambda_3 = \frac{\beta y^{\beta-1}}{p_2^\beta}$ also define a solution for conditions (i) – $(viii)$.

Finally, if $x_1 > 0$ and $x_2 > 0$, we have $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \frac{1}{p_1}$, $x_2 = \left(\frac{p_2}{\beta p_1}\right)^{\frac{1}{\beta-1}}$, $x_1 = \frac{y - \left(\frac{p_2}{\beta p_1}\right)^{\frac{1}{\beta-1}} p_2}{p_1} \geq 0$, or

$$\begin{aligned} y - \left(\frac{p_2}{\beta p_1}\right)^{\frac{1}{\beta-1}} p_2 &\geq 0 \\ y^{\beta-1} &\geq \frac{p_2^\beta}{\beta p_1} \end{aligned}$$

Hence, for $y^{\beta-1} \geq \frac{p_2^\beta}{\beta p_1}$, $x_1 = \frac{y - \left(\frac{p_2}{\beta p_1}\right)^{\frac{1}{\beta-1}} p_2}{p_1}$, $x_2 = \left(\frac{p_2}{\beta p_1}\right)^{\frac{1}{\beta-1}}$, $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \frac{1}{p_1}$ define a solution for conditions (i) – $(viii)$.

Note, however, that for $\beta > 1$, the indifference curves corresponding to u are concave and, hence, u is not quasi-concave. Hence, while necessary, the Kuhn-Tucker conditions are no longer sufficient for an optimum. The fact that the indifference curves are concave means that only $\left(x_1^* = 0; x_2^* = \frac{y}{p_2}\right)$ and $\left(x_1^* = \frac{y}{p_1}; x_2^* = 0\right)$ are candidates for an optimum. Comparing the utility Chris derives from these two bundles, we obtain:

$$u\left(x_1^* = \frac{y}{p_1}; x_2^* = 0\right) \gtrless u\left(x_1^* = 0; x_2^* = \frac{y}{p_2}\right)$$

iff

$$\begin{aligned} \frac{y}{p_1} &\gtrless \left(\frac{y}{p_2}\right)^\beta \\ \frac{p_2^\beta}{p_1} &\gtrless y^{\beta-1} \end{aligned}$$

Note that $y^{\beta-1} > \frac{p_2^\beta}{p_1}$ implies $y^{\beta-1} \geq \frac{p_2^\beta}{\beta p_1}$, the necessary condition for $\left(x_1^* = 0; x_2^* = \frac{y}{p_2}\right)$ to be an optimum. To summarize, if $\beta > 1$, Chris' demand is given by:

$$f_1(p_1; p_2; y) = \begin{cases} \frac{y}{p_1}, & y^{\beta-1} \leq \frac{p_2^\beta}{p_1} \\ 0, & y^{\beta-1} \geq \frac{p_2^\beta}{p_1} \end{cases}$$

$$f_2(p_1;p_2;y) = \begin{cases} 0, & y^{\beta-1} \leq \frac{p_2^\beta}{p_1} \\ \frac{y}{p_2}, & y^{\beta-1} \geq \frac{p_2^\beta}{p_1} \end{cases}$$

b) For $\beta = 1$, the income-offer curve is given in the lecture slides. For $\beta < 1$, the income offer curve is given by:

$$IOC = \begin{cases} x_2 = \left(\frac{\beta p_1}{p_2}\right)^{\frac{1}{1-\beta}}, & x_1 > 0 \\ x_1 = 0, & x_2 \leq \left(\frac{\beta p_1}{p_2}\right)^{\frac{1}{1-\beta}} \end{cases}$$

For $\beta > 1$, the income offer curve is:

$$IOC = \begin{cases} x_2 = 0, & x_1 \in \left[0; \left(\frac{p_2}{p_1}\right)^{\frac{\beta}{\beta-1}}\right] \\ x_1 = 0, & x_2 \geq \left(\frac{p_2}{p_1}\right)^{\frac{1}{\beta-1}} \end{cases}$$

c) For $\beta = 1$, the indirect utility function is given in the lecture slides. For $\beta < 1$,

$$v(p_1;p_2;y) = \begin{cases} \frac{y}{p_1} + (1-\beta)\left(\beta\frac{p_1}{p_2}\right)^{\frac{\beta}{1-\beta}}, & y^{1-\beta} \geq \frac{\beta p_1}{p_2^\beta} \\ \left(\frac{y}{p_2}\right)^\beta, & y^{1-\beta} < \frac{\beta p_1}{p_2^\beta} \end{cases}$$

For $\beta > 1$,

$$v(p_1;p_2;y) = \begin{cases} \frac{y}{p_1}, & y^{\beta-1} \leq \frac{p_2^\beta}{p_1} \\ \left(\frac{y}{p_2}\right)^\beta, & y^{\beta-1} > \frac{p_2^\beta}{p_1} \end{cases}$$

d) From the case $L = 2$, it is clear that all of these statements can only be true for $\beta < 1$.

(i) The optimization problem is now given by

$$\max_{\left\{x_1 \in \mathbb{R}; (x_2 \ldots x_L) \in \mathbb{R}_+^{L-1}\right\}} \left\{ x_1 + \sum_{l=2}^{L} (x_l)^\beta \mid \sum_{l=1}^{L} x_l p_l \leq y \right\}$$

$$\mathcal{L}(x_1 \ldots x_L; \lambda_1 \ldots \lambda_L) = x_1 + \sum_{l=2}^{L} (x_l)^\beta + \sum_{l=2}^{L} \lambda_l x_l + \lambda_1 \left(y - \sum_{l=1}^{L} x_l p_l \right)$$

The first-order conditions are:

$$(i)\ \frac{\partial \mathcal{L}(x_1 \ldots x_L; \lambda_1 \ldots \lambda_L)}{\partial x_1} = 1 - \lambda_1 p_1 = 0$$

$$(ii)\ \frac{\partial \mathcal{L}(x_1 \ldots x_L; \lambda_1 \ldots \lambda_L)}{\partial x_l} = \beta x_l^{\beta-1} + \lambda_l - \lambda_1 p_l = 0, l \in \{2 \ldots L\}$$

$$
\begin{aligned}
(iii)\, \lambda_l x_l &= 0, l \in \{2...L\} \\
(iv)\, \lambda_1 \left(y - \sum_{l=1}^{L} p_l x_l \right) &= 0
\end{aligned}
$$

$$
\begin{aligned}
(v)\, x_l &\geq 0, l \in \{2...L\} \\
(vi)\, y &\geq \sum_{l=1}^{L} p_l x_l \\
(viii)\, \lambda_l &\geq 0, l \in \{2...L\}
\end{aligned}
$$

Note that if $\lambda_1 = 0$, condition (i) cannot be satisfied. Hence, $\lambda_1 > 0$ and $y = \sum_{l=1}^{L} p_l x_l$ obtains. If $x_l = 0$ for some $l \in \{2...L\}$, then $\lim_{x_l \to 0} \beta x_l^{\beta-1} = \infty$ and the corresponding condition (ii) is not satisfied. Hence, $x_l > 0$ for all $l \in \{2...L\}$. It follows that the solution to the optimization problem is determined by the conditions

$$
MRS_{1,l} = \frac{x^{1-\beta}}{\beta} = \frac{p_1}{p_l} = \frac{1}{p_l}
$$

or $x_l^* = \left(\frac{\beta}{p_l}\right)^{\frac{1}{1-\beta}}$ for $l \in \{2...L\}$, which is independent of y.

(ii) The demand for good 1 is then given by

$$
x_1^* = \frac{y - \sum_{l=1}^{L} p_l \left(\frac{\beta}{p_l}\right)^{\frac{1}{1-\beta}}}{p_1} = y - \sum_{l=1}^{L} p_l \left(\frac{\beta}{p_l}\right)^{\frac{1}{1-\beta}}
$$

and is increasing in y.

(iii) The indirect utility function is given by:

$$
v(p_2...p_L; y) = y + (1-\beta) \sum_{l=1}^{L} \left(\frac{\beta}{p_l}\right)^{\frac{\beta}{1-\beta}}
$$

which is obviously linear in y and separable in y and p.

<u>Problem 2</u> *(Utility Maximization, Demand Theory, Constant Elasticity of Substitution)*

a) For $\rho = 1$, $u(x_1; x_2) = \alpha_1 x_1 + \alpha_2 x_2$. The equation of an indifference curve at utility level $\bar{u}$ is given by:

$$
x_2 = \frac{\bar{u}}{\alpha_2} - \frac{\alpha_1}{\alpha_2} x_1
$$

(See the lecture slides for a graph). These preferences mean that the two goods are perfect substitutes.

For $\rho \to 0$, first transform u by applying ln to get

$$\ln u(x_1; x_2) = \frac{\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)}{\rho}$$

and apply l'Hopital's rule:

$$\begin{aligned}
\lim_{\rho\to 0} \frac{\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)}{\rho} &= \lim_{\rho\to 0} \frac{[\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)]'_\rho}{(\rho)'_\rho} = \lim_{\rho\to 0} \frac{\alpha_1 x_1^\rho \ln x_1 + \alpha_2 x_2^\rho \ln x_2}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho} \\
&= \frac{\ln x_1^{\alpha_1} x_2^{\alpha_2}}{\alpha_1 + \alpha_2}
\end{aligned}$$

Hence, as ρ approaches 0, $u(\cdot)$ approaches a Cobb-Douglas utility function:

$$\lim_{\rho\to 0} u(x_1; x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$$

The corresponding indifference curve at a utility level $\bar{u}$ has the equation

$$x_2 = \frac{\bar{u}^{\frac{1}{\alpha_2}}}{x_1^{\frac{\alpha_1}{\alpha_2}}}$$

(See lecture slides for a graph). These preferences refer to two goods which are imperfect substitutes.

For $\rho \to -\infty$, apply again l'Hopital's rule to get:

$$\begin{aligned}
\lim_{\rho\to 0} \frac{\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)}{\rho} &= \lim_{\rho\to -\infty} \frac{\alpha_1 x_1^\rho \ln x_1 + \alpha_2 x_2^\rho \ln x_2}{\alpha_1 x_1^\rho + \alpha_2 x_2^\rho} \\
&= \lim_{\rho\to -\infty} \frac{\alpha_1 \left(\frac{x_1}{\min\{x_1;x_2\}}\right)^\rho \ln x_1 + \alpha_2 \left(\frac{x_2}{\min\{x_1;x_2\}}\right)^\rho \ln x_2}{\alpha_1 \left(\frac{x_1}{\min\{x_1;x_2\}}\right)^\rho + \alpha_2 \left(\frac{x_2}{\min\{x_1;x_2\}}\right)^\rho}
\end{aligned}$$

and note that if $x_1 > x_2$,

$$\begin{aligned}
\lim_{\rho\to -\infty} \left(\frac{x_1}{\min\{x_1; x_2\}}\right)^\rho &= 0 \\
\lim_{\rho\to -\infty} \left(\frac{x_2}{\min\{x_1; x_2\}}\right)^\rho &= 1
\end{aligned}$$

and symmetrically for $x_2 > x_1$. We thus obtain:

$$\lim_{\rho\to -\infty} \frac{\ln(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)}{\rho} = \ln \min\{x_1; x_2\}$$

Hence, as $\rho \to -\infty$, $u(\cdot)$ approaches the Leontieff function

$$u(x_1; x_2) = \min\{x_1; x_2\}$$

The corresponding indifference curve at a utility level $\bar{u}$ is given by the equation:

$$\begin{aligned} x_2 &= \bar{u}, \quad x_1 \geq \bar{u} \\ x_1 &= \bar{u}, \quad x_2 > \bar{u} \end{aligned}$$

(See lecture slides for a graph). These preferences imply that the two goods are perfect complements.

b) Utility maximization problems:

$$\max_{\left\{(x_1;x_2)\in\mathbb{R}_0^{2+}\right\}} \left\{(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} \mid x_1 p_1 + x_2 p_2 \leq y\right\}$$

Since preferences are locally non-satiated, the budget constraint is satisfied with equality.

Suppose first that $-\infty < \rho < 1$ and note that

$$MRS = \frac{x_1^{\rho-1}}{x_2^{\rho-1}} = \frac{x_2^{1-\rho}}{x_1^{1-\rho}}$$

and hence, as $x_1 \to 0$, $MRS \to \infty$ and as $x_2 \to 0$, $MRS \to 0$. It follows that for any values of p_1 and p_2,

$$\begin{aligned} \frac{p_1}{p_2} &> MRS\left(\frac{y}{p_1}; 0\right) \\ \frac{p_1}{p_2} &< MRS\left(0; \frac{y}{p_2}\right) \end{aligned}$$

and the problem has no corner solutions. Hence, the optimal consumption bundle satisfies:

$$\begin{aligned} MRS &= \frac{x_2^{1-\rho}}{x_1^{1-\rho}} = \frac{p_1}{p_2} \\ y &= p_1 x_1 + p_2 x_2 \end{aligned}$$

and we obtain

$$\begin{aligned} x_2 &= \left(\frac{p_1}{p_2}\right)^{\frac{1}{1-\rho}} x_1 \\ y &= p_1 x_1 + p_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{1-\rho}} x_1 \end{aligned}$$

$$x_1^* = f_1(p_1;p_2;y) = \frac{y}{p_1\left(1+\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}\right)}$$
$$x_2^* = f_2(p_1;p_2;y) = \frac{y}{p_2\left(1+\left(\frac{p_2}{p_1}\right)^{\frac{\rho}{1-\rho}}\right)}$$

Homogeneity of degree 0:

$$f_1(\lambda p_1;\lambda p_2;\lambda y) = \frac{\lambda y}{\lambda p_1\left(1+\left(\frac{\lambda p_1}{\lambda p_2}\right)^{\frac{\rho}{1-\rho}}\right)} = \frac{y}{p_1\left(1+\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}\right)}$$
$$f_2(\lambda p_1;\lambda p_2;\lambda y) = \frac{\lambda y}{\lambda p_2\left(1+\left(\frac{\lambda p_2}{\lambda p_1}\right)^{\frac{\rho}{1-\rho}}\right)} = \frac{y}{p_2\left(1+\left(\frac{p_2}{p_1}\right)^{\frac{\rho}{1-\rho}}\right)}$$

For $\rho = 1$, the solution is in the lecture slides. For $\rho > 1$, the indifference curves have an equation:

$$x_2 = (\bar{u} - x_1^\rho)^{\frac{1}{\rho}}$$

and are concave. It follows that the only possible solutions to the problem are $\left(x_1^* = \frac{y}{p_1}; x_2^* = 0\right)$ and $\left(x_1^* = 0; x_2^* = \frac{y}{p_2}\right)$. To check which of these obtains, we compute the utility of each:

$$u\left(x_1^* = \frac{y}{p_1}; x_2^* = 0\right) = \left(\frac{y}{p_1}\right)^\rho$$
$$u\left(x_1^* = 0; x_2^* = \frac{y}{p_2}\right) = \left(\frac{y}{p_2}\right)^\rho$$

and compare them.

$$u\left(x_1^* = \frac{y}{p_1}; x_2^* = 0\right) \gtrless u\left(x_1^* = 0; x_2^* = \frac{y}{p_2}\right)$$
$$\left(\frac{y}{p_1}\right)^\rho \gtrless \left(\frac{y}{p_2}\right)^\rho$$
$$p_2 \gtrless p_1$$

Hence,

$$f_1(p_1;p_2;y) = \begin{cases} \frac{y}{p_1}, & p_2 \geq p_1 \\ 0, & p_2 \leq p_1 \end{cases}$$
$$f_1(p_1;p_2;y) = \begin{cases} \frac{y}{p_2}, & p_2 \geq p_1 \\ 0, & p_2 \leq p_1 \end{cases}$$

Since $\frac{\lambda y}{\lambda p_i} = \frac{y}{p_i}$ for $i \in \{1;2\}$, the demand correspondences are homogeneous of degree 0.

Finally, for $\rho \to -\infty$, $u(x_1;x_2) = \min\{x_1;x_2\}$. We have that $MRS = 0$ if $x_1 > x_2$ and $MRS = \infty$ if $x_2 > x_1$. MRS is undefined (the IC is non-differentiable) at $x_1 = x_2$. Since $\frac{p_1}{p_2} \in (0;\infty)$, it follows that the optimum is given by

$$\begin{aligned} x_1^* &= f_1(p_1;p_2;y) = \frac{y}{p_1+p_2} \\ x_2^* &= f_2(p_1;p_2;y) = \frac{y}{p_1+p_2} \end{aligned}$$

Since $\frac{\lambda y}{\lambda(p_1+p_2)} = \frac{y}{p_1+p_2}$, the demand is homogeneous of degree 0.

c) For $-\infty < \rho < 1$, the indirect utility function is given by:

$$\begin{aligned} v(p_1;p_2;y) &= \left[\left[\frac{y}{p_1\left(1+\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}\right)}\right]^{\rho} + \left[\frac{y}{p_2\left(1+\left(\frac{p_2}{p_1}\right)^{\frac{\rho}{1-\rho}}\right)}\right]^{\rho}\right]^{\frac{1}{\rho}} \\ &= \left[y^{\rho}\left[\frac{p_2^{\frac{\rho}{1-\rho}}}{p_1\left(p_2^{\frac{\rho}{1-\rho}}+p_1^{\frac{\rho}{1-\rho}}\right)}\right]^{\rho} + \left[\frac{p_1^{\frac{\rho}{1-\rho}}}{p_2\left(p_1^{\frac{\rho}{1-\rho}}+p_2^{\frac{\rho}{1-\rho}}\right)}\right]^{\rho}\right]^{\frac{1}{\rho}} \\ &= \left[\frac{y}{p_1p_2\left(p_2^{\frac{\rho}{1-\rho}}+p_1^{\frac{\rho}{1-\rho}}\right)}\right]\left(p_2^{\frac{\rho}{1-\rho}}+p_1^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}} \\ &= y\frac{\left(p_1^{\frac{\rho}{1-\rho}}+p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}{p_1p_2} \end{aligned}$$

To see that $v(p_1;p_2;y)$ is non-increasing in p_1, differentiate w.r.t. p_1 to obtain:

$$\begin{aligned} \frac{\partial v(p_1;p_2;y)}{\partial p_1} &= \frac{y}{p_2}\frac{\left(p_1^{\frac{\rho}{1-\rho}}+p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}-1} p_1^{\frac{\rho}{1-\rho}} - \left(p_1^{\frac{\rho}{1-\rho}}+p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}{p_1^2} \\ &= -\frac{p_2^{\frac{\rho}{1-\rho}-1}}{p_1^2}y\left(p_1^{\frac{\rho}{1-\rho}}+p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}-1} < 0 \end{aligned}$$

The derivative w.r.t. p_2 is symmetric and also negative.

Homogeneity of degree 0:

$$v(\lambda p_1;\lambda p_2;\lambda y) = \lambda y\frac{\left((\lambda p_1)^{\frac{\rho}{1-\rho}}+(\lambda p_2)^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}{\lambda^2 p_1p_2}$$

$$= \lambda^2 y \frac{\left((p_1)^{\frac{\rho}{1-\rho}} + (p_2)^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}{\lambda^2 p_1 p_2}$$

$$= y \frac{\left(p_1^{\frac{\rho}{1-\rho}} + p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}{p_1 p_2} = v\left(p_1; p_2; y\right)$$

d) As $\rho \to 0$, Barbara's utility function becomes

$$u\left(x_1; x_2\right) = x_1^{\alpha_1} x_2^{\alpha_2}$$

The $MRS = \frac{\alpha_1 x_2}{\alpha_2 x_1}$ and we have $\lim_{x_1 \to 0} MRS = \infty$ and $\lim_{x_2 \to 0} MRS = 0$. Since $\frac{p_1}{p_2} \in (0; \infty)$, there are no corner solutions. The optimum is thus determined by

$$\begin{aligned} MRS &= \frac{\alpha_1 x_2}{\alpha_2 x_1} = \frac{p_1}{p_2} \\ x_1 p_1 + x_2 p_2 &= y \end{aligned}$$

It follows that

$$\begin{aligned} x_1^* &= f_1\left(p_1; p_2; y\right) = \frac{\alpha_1 y}{\left(\alpha_1 + \alpha_2\right) p_1} \\ x_2^* &= f_1\left(p_1; p_2; y\right) = \frac{\alpha_2 y}{\left(\alpha_1 + \alpha_2\right) p_2} \end{aligned}$$

To derive the income offer curve, note that

$$y = x_1 \frac{\left(\alpha_1 + \alpha_2\right)}{\alpha_1} p_1$$

and hence,

$$x_2 = \frac{\alpha_2}{\alpha_1} \frac{p_1}{p_2} x_1$$

gives the equation of the income offer curve.

The Engel curve for $p_1 = 1$ is given by:

$$x_1 = \frac{\alpha_1 y}{\left(\alpha_1 + \alpha_2\right)}$$

e) For $\rho \in (-\infty; 1)$,

$$\frac{f_1\left(p_1; p_2; y\right)}{f_2\left(p_1; p_2; y\right)} = \frac{y p_2 \left(1 + \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{1-\rho}}\right)}{y p_1 \left(1 + \left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}\right)} =$$

$$= \frac{yp_2\left(1+\left(\frac{p_2}{p_1}\right)^{\frac{\rho}{1-\rho}}\right)}{yp_1\left(1+\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}\right)} = \frac{\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{\rho-1}}}{\frac{p_1}{p_2}} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}$$

Hence,

$$\begin{aligned}
\xi_{12}(p_1;p_2;y) &= -\frac{\partial\left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}}{\partial\left[\frac{p_1}{p_2}\right]}\frac{\frac{p_1}{p_2}}{\left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}}} \\
&= -\frac{1}{\rho-1}\left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}-1}\frac{1}{\left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}-1}} = \frac{1}{1-\rho}
\end{aligned}$$

For $\rho \to 1$, $\xi_{12}(p_1;p_2;y) \to \infty$, i.e., the goods are perfect substitutes and any increase in the price of one of them shifts the entire demand towards the other. For $\rho \to 0$, $\xi_{12}(p_1;p_2;y) \to 1$, this is the Cobb-Douglas function with a constant elasticity of substitution equal to 1. For $\rho \to -\infty$, $\xi_{12}(p_1;p_2;y) \to 0$, this is the case of perfect complements in which the change in price of one of the goods does not change the consumption of the other.

Problem 3 *(Utility Maximization, Demand Theory, Perfect Complements)*

Anthony's preferences for goods 1...L are given by the utility function:

$$u(x_1...x_L) = \min\{x_1...x_L\}$$

He has an income of y and takes the prices $p_1...p_L$ as given.

a) Anthony's utility maximization problem:

$$\max_{\{(x_1...x_L)\in\mathbb{R}^L_+\}}\left\{\min\{x_1...x_L\} \mid \sum_{l=1}^{L} x_l p_l \le y\right\}$$

For $L = 2$, Anthony's indifference curves at a utility level $\bar{u}$ satisfy the equation $x_1 = \bar{u}$ if $x_2 \ge \bar{u}$, $x_2 = \bar{u}$ if $x_1 > \bar{u}$. Hence, the MRS satisfies $MRS(x) = \infty$ if $x_2 > x_1$ and $MRS(x) = 0$ if $x_1 > x_2$. At $x_1 = x_2$, the indifference curve is not differentiable and hence, the MRS is not defined. Since $\frac{p_1}{p_2} \in (0;\infty)$, it follows that in the optimum, $x_1^* = x_2^*$. Since preferences are monotone, the budget constraint is satisfied with equality and we obtain

$$\begin{aligned}
x_1p_1 + x_2p_2 &= y \\
x_1^* &= x_2^* = \frac{y}{p_1+p_2}
\end{aligned}$$

or, for $i \in \{1;2\}$, Anthony's Walrasian demand is:

$$f_i(p_1;p_2;y) = \frac{y}{p_1+p_2}$$

b) Barbara's Walrasian demand is given by

$$\begin{aligned} x_1^* &= f_1(p_1;p_2;y) = \frac{y}{p_1\left(1+\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}\right)} \\ x_2^* &= f_2(p_1;p_2;y) = \frac{y}{p_2\left(1+\left(\frac{p_2}{p_1}\right)^{\frac{\rho}{1-\rho}}\right)} \end{aligned}$$

For $i,\ j \in \{1,2\}$, $i \neq j$, we have:

$$\begin{aligned} \lim_{\rho\to-\infty} f_i(p_1;p_2;y) &= \lim_{\rho\to-\infty} \frac{y}{p_i\left(1+\left(\frac{p_i}{p_j}\right)^{\frac{\rho}{1-\rho}}\right)} = \frac{y}{p_i\left(1+\left(\frac{p_i}{p_j}\right)^{-1}\right)} \\ &= \frac{y}{p_i\left(1+\frac{p_j}{p_i}\right)} = \frac{y}{p_i+p_j} = \frac{y}{p_1+p_2} \end{aligned}$$

which exactly coincides with Anthony's Walrasian demand from part a).

c) For an arbitrary L, we have that the optimal bundle has to satisfy $x_l^* = const$ for all $l \in \{1...L\}$. Hence, from the budget constraint,

$$\sum_{l=1}^{L} x_l p_l = y$$

we obtain $x_l^* = \frac{y}{\sum_{l=1}^{L} p_l}$, or, for $l \in \{1...L\}$, Anthony's Walrasian demand is:

$$f_l(p;y) = \frac{y}{\sum_{l=1}^{L} p_l}$$

d) From the Walrasian demand for the first good, we obtain

$$y = x_1 \sum_{l=1}^{L} p_l$$

Hence,

$$x_l = x_1$$

for all $l \in \{2...L\}$ gives the equation of the income offer curve.

Problem 4 *(Utility Maximization, Kuhn-Tucker Method, Demand Theory)*

a) Note that

$$v(x_1;x_2;x_3) = [u(x_1;x_2;x_3)]^{\frac{1}{\alpha+\beta+\gamma}} = (x_1-b_1)^{\frac{\alpha}{\alpha+\beta+\gamma}}(x_2-b_2)^{\frac{\beta}{\alpha+\beta+\gamma}}(x_3-b_3)^{\frac{\gamma}{\alpha+\beta+\gamma}}$$

is a monotone transformation of $u(\cdot)$. Since the powers in v obviously sum up to 1, there is no loss of generality in assuming $\alpha+\beta+\gamma=1$.

Marianne's utility maximization problem:

$$\max_{\left\{(x_1;x_2;x_3)\in\mathbb{R}_0^{3+}\right\}} \left\{(x_1-b_1)^{\alpha}(x_2-b_2)^{\beta}(x_3-b_3)^{\gamma} \mid x_1p_1+x_2p_2+x_3p_3 \le y\right\}$$

In fact, the utility function is not well-defined for arbitrary powers α, β and $\gamma \in (0;1)$, unless $x_i \ge b_i$ for $i \in \{1,2,3\}$. Hence, we will look for a condition on p and y, for which Marianne chooses $x_i \ge b_i$ for $i \in \{1,2,3\}$.

At an interior solution, we have:

$$\begin{aligned} MRS_{12} &= \frac{\alpha(x_1-b_1)^{\alpha-1}(x_2-b_2)^{\beta}(x_3-b_3)^{\gamma}}{\beta(x_1-b_1)^{\alpha}(x_2-b_2)^{\beta-1}(x_3-b_3)^{\gamma}} = \frac{\alpha(x_2-b_2)}{\beta(x_1-b_1)} = \frac{p_1}{p_2} \\ MRS_{13} &= \frac{\alpha(x_3-b_3)}{\gamma(x_1-b_1)} = \frac{p_1}{p_3} \end{aligned}$$

and (since preferences are monotonic),

$$y = p_1x_1+p_2x_2+p_3x_3$$

Hence,

$$x_2 = \frac{p_1}{p_2}\frac{\beta}{\alpha}(x_1-b_1)+b_2$$

$$x_3 = \frac{p_1}{p_3}\frac{\gamma}{\alpha}(x_1-b_1)+b_3$$

and

$$p_1x_1+p_2x_2+p_3x_3 = p_1x_1+p_1\frac{\beta}{\alpha}(x_1-b_1)+p_2b_2+p_1\frac{\gamma}{\alpha}(x_1-b_1)+p_3b_3 = y$$

$$(x_1-b_1)\frac{p_1}{\alpha} = y-\sum_{i=1}^{3}p_ib_i$$

$x_1 \ge b_1$ obtains iff

$$y \ge \sum_{i=1}^{3}p_ib_i$$

Not as long as she consumes more than b_i of each of the goods.

b) From the computations in a), we obtain the Walrasian demand:

$$\begin{aligned} x_1^* &= f_1(p;y) = b_1 + \frac{\alpha}{p_1}\left[y - \sum_{j=1}^{3} p_j b_j\right] \\ x_2^* &= f_2(p;y) = b_2 + \frac{\beta}{p_2}\left[y - \sum_{j=1}^{3} p_j b_j\right] \\ x_3^* &= f_3(p;y) = b_3 + \frac{\beta}{p_3}\left[y - \sum_{j=1}^{3} p_j b_j\right] \end{aligned}$$

Since for $i \in \{1, 2, 3\}$,

$$\frac{1}{\lambda p_i}\left[\lambda y - \sum_{j=1}^{3} \lambda p_j b_j\right] = \frac{1}{p_i}\left[y - \sum_{j=1}^{3} p_j b_j\right]$$

we have that each of the f_i is homogeneous of degree 0.

c) Marianne's indirect utility function is

$$v(p;y) = \frac{\alpha^\alpha \beta^\beta \gamma^\gamma}{p_1^\alpha p_2^\beta p_3^\gamma}\left[y - \sum_{j=1}^{3} p_j b_j\right]$$

<ins>Problem 5</ins> *(Utility Maximization, Kuhn-Tucker Method, Demand Theory, Non-Satiation)*

a) Alice's indifference curves are circles centered around $(2;2)$. Alice's preferences do not satisfy any of the properties monotonicity, strict monotonicity and local non-satiation. All of these properties are violated at $x_1 = x_2 = 2$.

Alice's utility maximization problem:

$$\max_{\left\{(x_1;x_2)\in\mathbb{R}_0^{2+}\right\}} \left\{10 - (x_1-2)^2 - (x_2-2)^2 \mid p_1x_1 + p_2x_2 \le y\right\}$$

Write down the Lagrangian:

$$\mathcal{L}(x_1;x_2;\lambda_1;\lambda_2;\lambda_3) = 10 - (x_1-2)^2 - (x_2-2)^2 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3(y - x_1p_1 - x_2p_2)$$

The f.o.c. are

$$\begin{aligned} -2(x_1-2) + \lambda_1 - \lambda_3 p_1 &= 0 \\ -2(x_2-2) + \lambda_2 - \lambda_3 p_2 &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_i x_i &= 0, i \in \{1,2\} \\ \lambda_3 (y - x_1 p_1 - x_2 p_2) &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_i &\geq 0, i \in \{1...3\} \\ x_i &\geq 0, i \in \{1,2\} \\ y - x_1 p_1 - x_2 p_2 &\geq 0 \end{aligned}$$

(i) $x_1 = 0$ corresponds to $\lambda_1 \geq 0$ and

$$\lambda_3 = \frac{\lambda_1 + 4}{p_1} > 0$$

It follows that the budget constraint is satisfied with equality and hence, $x_2 = \frac{y}{p_2}$ and $\lambda_2 = 0$. We thus obtain

$$\begin{aligned} 4 - 2\frac{y}{p_2} - (\lambda_1 + 4)\frac{p_2}{p_1} &= 0 \\ \lambda_1 &= 4\left(\frac{p_1}{p_2} - 1\right) - 2\frac{yp_1}{p_2^2} \end{aligned}$$

Since $\lambda_1 \geq 0$, the condition which has to hold in order for $x_1 = 0$ to be optimal is

$$\begin{aligned} 4\left(\frac{p_1}{p_2} - 1\right) - 2\frac{yp_1}{p_2^2} &\geq 0 \\ 2\left(1 - \frac{p_2}{p_1}\right) &\geq \frac{y}{p_2} \end{aligned}$$

(ii) Symmetrically, for $x_2 = 0$, the condition is:

$$2\left(1 - \frac{p_1}{p_2}\right) \geq \frac{y}{p_1}$$

(iii) For $\lambda_3 = 0$ to hold, we need:

$$\begin{aligned} x_1 - 2 &= \frac{\lambda_1}{2} \\ x_2 - 2 &= \frac{\lambda_2}{2} \end{aligned}$$

or $x_1 \geq 2$ and $x_2 \geq 2$. Since for $x_1 > 0$ and $x_2 > 0$, $\lambda_1 = \lambda_2 = 0$, we have $x_1 = x_2 = 2$. Finally, from the budget constraint, we obtain:

$$2p_1 + 2p_2 < y$$

to be the condition under which she will spend less than y on pastries.

b) In a), we derived the corner solutions as well as the solution for the case in which the budget constraint is not binding.

Hence, we just need to derive the interior solution, $x_1 > 0$, $x_2 > 0$, for which the budget constraint is binding. It follows that $\lambda_1 = \lambda_2 = 0$ and $x_1 p_1 + x_2 p_2 = y$. We obtain:

$$\begin{aligned} \frac{\lambda_3}{2} &= \frac{2 - x_1}{p_1} \\ \frac{\lambda_3}{2} &= \frac{2 - x_2}{p_2} \end{aligned}$$

and hence, $x_1 \leq 2$, $x_2 \leq 2$ and

$$\frac{2 - x_1}{p_1} = \frac{2 - x_2}{p_2}$$

$$x_2 = \frac{2(p_1 - p_2) + x_1 p_2}{p_1}$$

and from the budget constraint

$$x_1 = \frac{y p_1 - 2(p_1 - p_2) p_2}{(p_1^2 + p_2^2)}$$

Obviously, $x_1 \geq 0$ iff

$$\frac{y}{p_2} \geq 2\left(1 - \frac{p_2}{p_1}\right)$$

and $x_1 \leq 2$ iff

$$\begin{aligned} \frac{y p_1 - 2(p_1 - p_2) p_2}{(p_1^2 + p_2^2)} &\leq 2 \\ 2p_1 + 2p_2 &\geq y \end{aligned}$$

For x_2, we obtain:

$$\begin{aligned} x_2 &= \frac{2(p_1 - p_2) + \frac{y p_1 - 2(p_1 - p_2) p_2}{(p_1^2 + p_2^2)} p_2}{p_1} \\ &= \frac{\frac{2(p_1 - p_2)(p_1^2 + p_2^2) - 2(p_1 - p_2) p_2^2}{(p_1^2 + p_2^2)} + \frac{y p_1 p_2}{(p_1^2 + p_2^2)}}{p_1} \\ &= \frac{y p_2 - 2(p_2 - p_1) p_1}{(p_1^2 + p_2^2)} \end{aligned}$$

$x_2 \geq 0$ iff

$$\frac{y}{p_1} \geq 2\left(1 - \frac{p_1}{p_2}\right)$$

and $x_2 \leq 2$ iff $2p_1 + 2p_2 \leq y$.

To summarize, Alice's Walrasian demand for cookies and cupcakes is given by:

$$f_1(p_1; p_2; y) = \begin{cases} 0, & p_1 > p_2, y < 2\left(p_2 - \frac{p_2^2}{p_1}\right) \\ \frac{y}{p_1}, & p_1 < p_2, y < 2\left(p_1 - \frac{p_1^2}{p_2}\right) \\ \frac{yp_1 - 2(p_1 - p_2)p_2}{\left(p_1^2 + p_2^2\right)}, & y \in \left[\max\left\{2\left(p_2 - \frac{p_2^2}{p_1}\right); 2\left(p_1 - \frac{p_1^2}{p_2}\right)\right\}; 2p_1 + 2p_2\right] \\ 2, & y > 2p_1 + 2p_2 \end{cases}$$

$$f_2(p_1; p_2; y) = \begin{cases} 0, & p_2 > p_1, y < 2\left(p_1 - \frac{p_1^2}{p_2}\right) \\ \frac{y}{p_2}, & p_2 < p_1, y < 2\left(p_2 - \frac{p_2^2}{p_1}\right) \\ \frac{yp_2 - 2(p_2 - p_1)p_1}{\left(p_1^2 + p_2^2\right)}, & y \in \left[\max\left\{2\left(p_2 - \frac{p_2^2}{p_1}\right); 2\left(p_1 - \frac{p_1^2}{p_2}\right)\right\}; 2p_1 + 2p_2\right] \\ 2, & y > 2p_1 + 2p_2 \end{cases}$$

It satisfies homogeneity of degree 0, convex-valuedness and uniqueness, but violates the Walras' Law for $y > 2p_1 + 2p_2$.

c) Alice's indirect utility function:

$$v(p_1; p_2; y) = \begin{cases} 2 - \left(\frac{y}{p_1}\right)^2 + 4\frac{y}{p_1}, & p_2 > p_1, y < 2\left(p_1 - \frac{p_1^2}{p_2}\right) \\ 2 - \left(\frac{y}{p_2}\right)^2 + 4\frac{y}{p_2}, & p_2 < p_1, y < 2\left(p_2 - \frac{p_2^2}{p_1}\right) \\ 10 - \frac{(y - 2(p_1 + p_2))^2}{p_1^2 + p_2^2}, & y \in \left[\max\left\{2\left(p_2 - \frac{p_2^2}{p_1}\right); 2\left(p_1 - \frac{p_1^2}{p_2}\right)\right\}; 2p_1 + 2p_2\right] \\ 10, & y > 2p_1 + 2p_2 \end{cases}$$

since

$$\begin{aligned} & 10 - \left(\frac{yp_1 - 2(p_1 - p_2)p_2}{(p_1^2 + p_2^2)} - 2\right)^2 - \left(\frac{yp_2 - 2(p_2 - p_1)p_1}{(p_1^2 + p_2^2)} - 2\right)^2 \\ = & \ 10 - \frac{p_1^2}{(p_1^2 + p_2^2)^2}(y - 2(p_1 + p_2))^2 - \frac{p_2^2}{(p_1^2 + p_2^2)^2}(y - 2(p_1 + p_2))^2 \\ = & \ 10 - \frac{(y - 2(p_1 + p_2))^2}{p_1^2 + p_2^2} \end{aligned}$$

It is not strictly increasing in y.

Problem 6 *(Utility Maximization, Demand Theory, Convexity)*

a) The equation of Peter's indifference curve at utility level $\bar{u}$ is given by

$$x_2^2 = \sqrt{\bar{u} - \alpha x_1^2}$$

which is a concave function.

Hence, Peter's preferences do not satisfy strict or weak convexity. Since x^2 is a convex function, we have that for any $\lambda \in (0;1)$,

$$\begin{aligned} & u\left(\lambda\left(x_1;x_2\right)+(1-\lambda)\left(x_1';x_2'\right)\right) \\ = & \alpha\left[\lambda x_1+(1-\lambda)x_1'\right]^2+\left[\lambda x_2+(1-\lambda)x_2'\right]^2 \\ \leq & \alpha\lambda x_1^2+\alpha(1-\lambda)x_1'^2+\lambda x_2^2+(1-\lambda)x_2'^2 \\ = & \lambda u\left(x_1;x_2\right)+(1-\lambda)u\left(x_1';x_2'\right) \end{aligned}$$

Peter's utility maximization problem:

$$\max_{\left\{(x_1;x_2)\in\mathbb{R}_0^{2+}\right\}}\left\{\alpha x_1^2+x_2^2 \mid p_1x_1+p_2x_2\leq y\right\}$$

b) Since the indifference curves are concave, the Peter's optimization problem can only have corner solutions, either $(x_1^*;x_2^*)=\left(\frac{y}{p_1};0\right)$ or $(x_1^*;x_2^*)=\left(0;\frac{y}{p_2}\right)$.

To check which of these is the solution, we compare

$$\begin{aligned} u\left(\frac{y}{p_1};0\right) & \gtrless u\left(0;\frac{y}{p_2}\right) \\ \alpha\left(\frac{y}{p_1}\right)^2 & \gtrless \left(\frac{y}{p_2}\right)^2 \\ \sqrt{\alpha} & \gtrless \frac{p_1}{p_2} \end{aligned}$$

Hence, Peter's Walrasian demand is given by

$$f_1\left(p_1;p_2;y\right)=\begin{cases}\frac{y}{p_1}, & \frac{p_1}{p_2}\leq\sqrt{\alpha}\\ 0, & \frac{p_1}{p_2}\geq\sqrt{\alpha}\end{cases}$$

$$f_1\left(p_1;p_2;y\right)=\begin{cases}0, & \frac{p_1}{p_2}\leq\sqrt{\alpha}\\ \frac{y}{p_2}, & \frac{p_1}{p_2}\geq\sqrt{\alpha}\end{cases}$$

His demand is not single-valued, in particular, when $\frac{p_1}{p_2}=\sqrt{\alpha}$. It is also not convex-valued, since at $\frac{p_1}{p_2}=\sqrt{\alpha}$, $f_1\left(p_1;p_2;y\right)\in\left\{0;\frac{y}{p_1}\right\}$, but no $x_1\in\left(0;\frac{y}{p_1}\right)$ is optimal. Peter never plans a vacation including both activities.

c) Peter's income-offer curve is given by $x_1=0$ if $\frac{p_1}{p_2}>\sqrt{\alpha}$, $x_2=0$ if $\frac{p_1}{p_2}<\sqrt{\alpha}$ and the two lines $x_1=0$, $x_2=0$ when $\frac{p_1}{p_2}=\sqrt{\alpha}$. His Engel curve for good 1 at $p_1=1$ is $x_1=y$ if $\frac{p_1}{p_2}<\sqrt{\alpha}$, $x_1=0$ if $\frac{p_1}{p_2}>\sqrt{\alpha}$ and the two lines $x_1=y$ and $x_1=0$ if $\frac{p_1}{p_2}=\sqrt{\alpha}$.

d) Peter's indirect utility function is given by

$$v\left(p_1;p_2;y\right)=\begin{cases}\alpha\frac{y^2}{p_1^2}, & \frac{p_1}{p_2}\leq\sqrt{\alpha}\\ \frac{y_2^2}{p_2^2}, & \frac{p_1}{p_2}\geq\sqrt{\alpha}\end{cases}$$

Quasi-convexity of v is equivalent to

$$v\left(\lambda\left(p_1;p_2;y\right)+\left(1-\lambda\right)\left(p_1';p_2';y'\right)\right)\leq\max\left\{v\left(p_1;p_2;y\right);v\left(p_1';p_2';y'\right)\right\}$$

If $\frac{p_1}{p_2}\leq\sqrt{\alpha}$ and $\frac{p_1'}{p_2'}\leq\sqrt{\alpha}$, then

$$\begin{aligned}\lambda p_1 &\leq \lambda\sqrt{\alpha}p_2\\ \left(1-\lambda\right)p_1' &\leq \left(1-\lambda\right)\sqrt{\alpha}p_2'\end{aligned}$$

and

$$\lambda p_1+\left(1-\lambda\right)p_1'\leq\lambda\sqrt{\alpha}p_2+\left(1-\lambda\right)\sqrt{\alpha}p_2'$$

or, $\frac{\lambda p_1+(1-\lambda)p_1'}{\lambda p_2+(1-\lambda)p_2'}\leq\sqrt{\alpha}$. Hence,

$$\begin{aligned}&v\left(\lambda\left(p_1;p_2;y\right)+\left(1-\lambda\right)\left(p_1';p_2';y'\right)\right)\\ =\;&\alpha\frac{\left[\lambda y+\left(1-\lambda\right)y'\right]^2}{\left(\lambda p_1+\left(1-\lambda\right)p_1'\right)^2}\leq\frac{\alpha\max\left\{y^2;\left(y'\right)^2\right\}}{\max\left\{p_1^2;\left(p_1'\right)^2\right\}}\leq\max\left\{\frac{\alpha y^2}{p_1^2};\frac{\alpha\left(y'\right)^2}{\left(p_1'\right)^2}\right\}\end{aligned}$$

where the second inequality follows from the fact that both the denominator and the numerator are monotonic, and hence, quasi-convex functions and the last inequality follows from the fact that

$$\frac{1}{p_i^2}\geq\frac{1}{\max\left\{p_1^2;\left(p_1'\right)^2\right\}}$$

for $i\in\{1,2\}$, thus proving quasi-convexity in this case. Proceed similarly for the case $\frac{p_1}{p_2}\geq\sqrt{\alpha}$ and $\frac{p_1'}{p_2'}\geq\sqrt{\alpha}$. If $\frac{p_1}{p_2}>\sqrt{\alpha}$ and $\frac{p_1'}{p_2'}<\sqrt{\alpha}$, then whether or not

$$\frac{\lambda p_1+\left(1-\lambda\right)p_1'}{\lambda p_2+\left(1-\lambda\right)p_2'}\leq\sqrt{\alpha}$$

obtains depends on λ. Suppose the condition is satisfied, then

$$\begin{aligned}&v\left(\lambda\left(p_1;p_2;y\right)+\left(1-\lambda\right)\left(p_1';p_2';y'\right)\right)\\ =\;&\alpha\frac{\left[\lambda y+\left(1-\lambda\right)y'\right]^2}{\left(\lambda p_1+\left(1-\lambda\right)p_1'\right)^2}\leq\frac{\alpha\max\left\{y^2;\left(y'\right)^2\right\}}{\max\left\{p_1^2;\left(p_1'\right)^2\right\}}\leq\max\left\{\frac{\alpha y^2}{p_1^2};\frac{\alpha\left(y'\right)^2}{\left(p_1'\right)^2}\right\}\\ <\;&\max\left\{\frac{y^2}{p_2^2};\frac{\alpha\left(y'\right)^2}{\left(p_1'\right)^2}\right\}\end{aligned}$$

where the last inequality follows from the fact that $\frac{p_1}{p_2} > \sqrt{\alpha}$ and thus, $\frac{y^2}{p_2^2} > \alpha\frac{y^2}{p_1^2}$, and we obtain quasi-convexity. Proceed similarly for the case $\frac{p_1}{p_2} < \sqrt{\alpha}$ and $\frac{p_1'}{p_2'} > \sqrt{\alpha}$.