Microeconomics – solutions to problem set 4

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Problem 1 *(Expenditure Minimization, Kuhn-Tucker Method, Duality, Quasi-Linear Utility)*

a) Chris' expenditure minimization problem:

$$\max_{\left\{(x_1 \ldots x_L) \in \mathbb{R}_+^L\right\}} \left\{ \sum_{l=1}^{L} x_l p_l \mid x_1 + \sum_{l=2}^{L} (x_l)^\beta \geq \bar{u} \right\}$$

$$\mathcal{L}(x_1; x_2; \lambda_1; \lambda_2; \lambda_3) = -p_1 x_1 - p_2 x_2 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 \left(x_1 + x_2^\beta - \bar{u} \right)$$

The first-order conditions are:

$$\begin{aligned} (i)\ \frac{\partial \mathcal{L}(x_1; x_2; \lambda_1; \lambda_2; \lambda_3)}{\partial x_1} &= -p_1 + \lambda_1 + \lambda_3 = 0 \\ (ii)\ \frac{\partial \mathcal{L}(x_1; x_2; \lambda_1; \lambda_2; \lambda_3)}{\partial x_2} &= -p_2 + \lambda_2 + \lambda_3 \beta x_2^{\beta-1} = 0 \end{aligned}$$

$$\begin{aligned} (iii)\ \lambda_1 x_1 &= 0 \\ (iv)\ \lambda_2 x_2 &= 0 \\ (v)\ \lambda_3 \left(x_1 + x_2^\beta - \bar{u} \right) &= 0 \end{aligned}$$

$$\begin{aligned} (vi)\ x_1 &\geq 0, x_2 \geq 0 \\ (vii)\ x_1 + x_2^\beta &\geq \bar{u} \\ (viii)\ \lambda_1, \lambda_2, \lambda_3 &\geq 0 \end{aligned}$$

Note that if $\lambda_3 = 0$, we have $\lambda_1 = p_1 > 0$ and $\lambda_2 = p_2 > 0$, hence, $x_1 = x_2 = 0$, or $u(x_1; x_2) = 0 < \bar{u}$, a contradiction. Hence, $\lambda_3 > 0$ and the utility constraint always holds with equality:

$$x_1 + x_2^\beta = \bar{u}$$

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If $\beta = 1$, $MRS = 1$ and the Hicks' demand is given by

$$h_1(p_1;p_2;\bar{u}) = \begin{cases} 0, & \frac{p_1}{p_2} > 1 \\ [0;\bar{u}], & \frac{p_1}{p_2} = 1 \\ \bar{u}, & \frac{p_1}{p_2} < 1 \end{cases}$$

$$h_2(p_1;p_2;\bar{u}) = \begin{cases} 0, & \frac{p_1}{p_2} < 1 \\ [0;\bar{u}], & \frac{p_1}{p_2} = 1 \\ \bar{u}, & \frac{p_1}{p_2} > 1 \end{cases}$$

Let $\beta < 1$ and suppose that $x_2 = 0$, and hence, $x_1 = \bar{u} > 0$. It follows that $\lambda_1 = 0$ and $\lambda_3 = p_1$. Since $\beta < 1$, $\lim_{x_2 \to 0} \frac{\beta}{x_2^{1-\beta}} = \infty$, hence there is no λ_2 which would satisfy condition (ii).

If $x_1 = 0$, then $x_2 = \bar{u}^{\frac{1}{\beta}} > 0$. It follows that $\lambda_2 = 0$, $\lambda_3 = p_2 \frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta}$

$$\lambda_1 = p_1 - \lambda_3 = p_1 - p_2 \frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta} \geq 0$$

if $\frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta} \leq \frac{p_1}{p_2}$. Hence, for $\frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta} \leq \frac{p_1}{p_2}$, $x_1 = 0$, $x_2 = \bar{u}^{\frac{1}{\beta}}$, $\lambda_1 = p_1 - p_2 \frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta}$, $\lambda_2 = 0$ and $\lambda_3 = p_2 \frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta}$ define a solution for conditions (i) – $(viii)$.

Finally, if $x_1 > 0$ and $x_2 > 0$, we have $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = p_1$, $x_2 = \left(\frac{\beta p_1}{p_2}\right)^{\frac{1}{1-\beta}}$, $x_1 = \bar{u} - \left(\frac{\beta p_1}{p_2}\right)^{\frac{\beta}{1-\beta}} \geq 0$, or

$$\begin{aligned} \bar{u} - \left(\frac{\beta p_1}{p_2}\right)^{\frac{\beta}{1-\beta}} &\geq 0 \\ \frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta} &\geq \frac{p_1}{p_2} \end{aligned}$$

Hence, for $\frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta} \geq \frac{p_1}{p_2}$, $x_1 = \bar{u} - \left(\frac{\beta p_1}{p_2}\right)^{\frac{\beta}{1-\beta}}$, $x_2 = \left(\frac{\beta p_1}{p_2}\right)^{\frac{1}{1-\beta}}$, $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = p_1$ define a solution for conditions (i) – $(viii)$. Note that when $\beta < 1$, the function $x_1 + x_2^{\beta}$ is concave (and hence, quasi-concave). Hence, the Kuhn-Tucker conditions are necessary and sufficient for the existence of an optimum. To summarize, Chris' demand is given by:

$$h_1(p_1;p_2;y) = \begin{cases} \bar{u} - \left(\frac{\beta p_1}{p_2}\right)^{\frac{\beta}{1-\beta}}, & \frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta} \geq \frac{p_1}{p_2} \\ 0, & \frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta} < \frac{p_1}{p_2} \end{cases}$$

$$h_2(p_1;p_2;y) = \begin{cases} \left(\frac{\beta p_1}{p_2}\right)^{\frac{1}{1-\beta}}, & \frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta} \geq \frac{p_1}{p_2} \\ \bar{u}^{\frac{1}{\beta}}, & \frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta} \geq \frac{p_1}{p_2} \end{cases}$$

Let $\beta > 1$. The indifference curves corresponding to u are concave and, hence, u is not quasi-concave. Hence, while necessary, the Kuhn-Tucker conditions are no longer sufficient for an optimum. The fact that the indifference curves are concave means that only $\left(x_1^* = 0; x_2^* = \bar{u}^{\frac{1}{\beta}}\right)$ and $(x_1^* = \bar{u}; x_2^* = 0)$ are candidates for an optimum. Comparing the expenditures for these two bundles, we obtain:

$$e(x_1^* = \bar{u}; x_2^* = 0) \gtrless e\left(x_1^* = 0; x_2^* = \bar{u}^{\frac{1}{\beta}}\right)$$

iff

$$\begin{aligned} \bar{u}p_1 &\gtrless \bar{u}^{\frac{1}{\beta}} p_2 \\ \bar{u}^{\frac{\beta-1}{\beta}} &\gtrless \frac{p_2}{p_1} \end{aligned}$$

To summarize, if $\beta > 1$, Chris' demand is given by:

$$h_1(p_1;p_2;y) = \begin{cases} \bar{u}, & \bar{u}^{\frac{\beta-1}{\beta}} \leq \frac{p_2}{p_1} \\ 0, & \bar{u}^{\frac{\beta-1}{\beta}} \geq \frac{p_2}{p_1} \end{cases}$$

$$h_2(p_1;p_2;y) = \begin{cases} 0, & \bar{u}^{\frac{\beta-1}{\beta}} \leq \frac{p_2}{p_1} \\ \bar{u}^{\frac{1}{\beta}}, & \bar{u}^{\frac{\beta-1}{\beta}} \geq \frac{p_2}{p_1} \end{cases}$$

b) For $\beta = 1$, the expenditure function is:

$$e(p_1;p_2;\bar{u}) = \begin{cases} \bar{u}p_2, & \frac{p_1}{p_2} \geq 1 \\ \bar{u}p_1, & \frac{p_1}{p_2} \leq 1 \end{cases}$$

For $\beta < 1$, the expenditure function is:

$$e(p_1;p_2;y) = \begin{cases} \bar{u}p_1 - (1-\beta)\frac{\beta^{\frac{\beta}{1-\beta}} p_1^{\frac{1}{1-\beta}}}{p_2^{\frac{\beta}{1-\beta}}}, & \frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta} \geq \frac{p_1}{p_2} \\ \bar{u}^{\frac{1}{\beta}} p_2, & \frac{\bar{u}^{\frac{1-\beta}{\beta}}}{\beta} < \frac{p_1}{p_2} \end{cases}$$

For $\beta > 1$, the expenditure function is:

$$e(p_1; p_2; y) = \begin{cases} \bar{u}p_1, & \bar{u}^{\frac{\beta-1}{\beta}} \leq \frac{p_2}{p_1} \\ \bar{u}^{\frac{1}{\beta}}p_2, & \bar{u}^{\frac{\beta-1}{\beta}} \geq \frac{p_2}{p_1} \end{cases}$$

c)

$$\max_{\left\{(x_1...x_L)\in\mathbb{R}^L_+\right\}} \left\{\sum_{l=1}^{L} x_l p_l \mid x_1 + \sum_{l=2}^{L} (x_l)^\beta \geq \bar{u}\right\}$$

$$\mathcal{L}(x_1...x_L; \lambda_1...\lambda_L) = -\sum_{l=1}^{L} x_l p_l + \sum_{l=2}^{L} \lambda_l x_l + \lambda_1 \left(x_1 + x_2^\beta - \bar{u}\right)$$

The first-order conditions are:

$$\begin{aligned} (i)\ \frac{\partial\mathcal{L}(x_1...x_L; \lambda_1...\lambda_L)}{\partial x_1} &= -p_1 + \lambda_1 = 0 \\ (ii)\ \frac{\partial\mathcal{L}(x_1...x_L; \lambda_1...\lambda_L)}{\partial x_l} &= -p_l + \lambda_l + \lambda_1\beta x_l^{\beta-1} = 0 \end{aligned}$$

$$\begin{aligned} (iii)\ \lambda_l x_l &= 0, l \in \{2...L\} \\ (iv)\ \lambda_1 \left(x_1 + \sum_{l=2}^{L} (x_l)^\beta - \bar{u}\right) &= 0 \end{aligned}$$

$$\begin{aligned} (v)\ x_l &\geq 0, l \in \{2...L\} \\ (vi)\ x_1 + \sum_{l=2}^{L} (x_l)^\beta &\geq \bar{u} \\ (vii)\ \lambda_1...\lambda_L &\geq 0 \end{aligned}$$

(i) Note that $\lambda_1 = p_1 = 1 \neq 0$, hence, $x_1 + \sum_{l=2}^{L} (x_l)^\beta = \bar{u}$ always holds. Furthermore, since $\lim_{x_l\to 0} \beta x_l^{\beta-1} = \infty$, we have that $x_l \neq 0$ for all $l \in \{2...L\}$ and, thus, $\lambda_l = 0$ for all $l \in \{2...L\}$. It follows that

$$\begin{aligned} p_1\beta x_l^{\beta-1} &= \beta x_l^{\beta-1} = p_l \\ h_l(p; \bar{u}) &= \left(\frac{\beta}{p_l}\right)^{\frac{1}{1-\beta}} \end{aligned}$$

for $l \in \{2...L\}$, which is independent of $\bar{u}$.

$$h_1(p; \bar{u}) = \bar{u} - \sum_{l=2}^{L} \left(\frac{\beta}{p_l}\right)^{\frac{\beta}{1-\beta}}$$

(ii) The expenditure function is

$$
\begin{aligned}
e(p;\bar{u}) &= \bar{u} - \sum_{l=2}^{L} \left(\frac{\beta}{p_l}\right)^{\frac{\beta}{1-\beta}} + \sum_{l=2}^{L} \left(\frac{\beta}{p_l}\right)^{\frac{1}{1-\beta}} p_l \\
&= \bar{u} - \sum_{l=2}^{L} \frac{\beta^{\frac{\beta}{1-\beta}}}{p_l^{\frac{\beta}{1-\beta}}} + \sum_{l=2}^{L} \frac{\beta^{\frac{1}{1-\beta}}}{p_l^{\frac{\beta}{1-\beta}}} \\
&= \bar{u} - (1-\beta) \sum_{l=2}^{L} \frac{\beta^{\frac{\beta}{1-\beta}}}{p_l^{\frac{\beta}{1-\beta}}}
\end{aligned}
$$

<u>Problem 2</u> *(Demand and duality, Cobb-Douglas utility function)*

a) The conditions for utility maximizations are given by

$$
\begin{aligned}
MRS &= \frac{x_2}{2x_1} = \frac{p_1}{p_2} \\
x_1 p_1 + x_2 p_2 &= y
\end{aligned}
$$

Hence, the Walrasian demand is:

$$
\begin{aligned}
f_1(p_1;p_2;y) &= \frac{y}{3p_1} \\
f_2(p_1;p_2;y) &= \frac{2y}{3p_2}
\end{aligned}
$$

and the indirect utility function $v(p_1;p_2;y)$ is

$$
v(p_1;p_2;y) = \frac{4y^3}{27p_1p_2^2}
$$

b) We have:

$$
\begin{aligned}
v(p_1;p_2;e(p_1;p_2;\bar{u})) &= \bar{u} \\
\frac{4e^3(p_1;p_2;\bar{u})}{27p_1p_2^2} &= \bar{u}
\end{aligned}
$$

The expenditure function is then given by

$$
e(p_1;p_2;\bar{u}) = 3\sqrt[3]{\frac{\bar{u}p_1p_2^2}{4}}
$$

$e(p_1;p_2;\bar{u})$.

c) The cost minimization problem:

$$\min_{\left\{(x_1;x_2)\in\mathbb{R}_0^{2+}\right\}} \left\{p_1x_1+p_2x_2 \mid u(x)=x_1x_2^2\geq\bar{u}\right\}$$

In the optimum,

$$\begin{aligned} MRS &= \frac{x_2}{2x_1}=\frac{p_1}{p_2} \\ x_1x_2^2 &= \bar{u} \end{aligned}$$

The Hicksian demand functions.are:

$$\begin{aligned} h_1(p_1;p_2;\bar{u}) &= \sqrt[3]{\bar{u}\frac{p_2^2}{4p_1^2}} \\ h_2(p_1;p_2;\bar{u}) &= \sqrt[3]{\frac{2\bar{u}p_1}{p_2}} \end{aligned}$$

d) To see that Shepard's lemma holds,

$$\begin{aligned} \frac{\partial e(p_1;p_2;\bar{u})}{\partial p_1} &= \left(3\sqrt[3]{\frac{\bar{u}p_1p_2^2}{4}}\right)'_{p_1}=\frac{3}{3}\left(\frac{\bar{u}^{\frac{1}{3}}p_1^{-\frac{2}{3}}p_2^{\frac{2}{3}}}{4^{\frac{1}{3}}}\right)=\sqrt[3]{\frac{\bar{u}p_2^2}{4p_1^2}}=h_1(p_1;p_2;\bar{u}) \\ \frac{\partial e(p_1;p_2;\bar{u})}{\partial p_2} &= \left(3\sqrt[3]{\frac{\bar{u}p_1p_2^2}{4}}\right)'_{p_2}=\frac{3\cdot 2}{3}\frac{\bar{u}^{\frac{1}{3}}p_1^{\frac{1}{3}}p_2^{-\frac{1}{3}}}{4^{\frac{1}{3}}}=\sqrt[3]{\frac{2\bar{u}p_1}{p_2}}=h_2(p_1;p_2;\bar{u}) \end{aligned}$$

.

e) Homogeneity of degree 0 in prices:

$$\begin{aligned} h_1(\lambda p_1;\lambda p_2;\bar{u}) &= \sqrt[3]{\bar{u}\frac{\lambda^2p_2^2}{4\lambda^2p_1^2}}=\sqrt[3]{\bar{u}\frac{p_2^2}{4p_1^2}}=h_1(p_1;p_2;\bar{u}) \\ h_2(\lambda p_1;\lambda p_2;\bar{u}) &= \sqrt[3]{\frac{2\bar{u}\lambda p_1}{\lambda p_2}}=\sqrt[3]{\frac{2\bar{u}p_1}{p_2}}=h_2(p_1;p_2;\bar{u}) \end{aligned}$$

The fact that $h_i(p_1;p_2;\bar{u})$ is decreasing in p_i is obvious.

f) Show that the expenditure function is:

*strictly increasing in p_1 and p_2 — obvious; *strictly increasing in $\bar{u}$ — obvious; *homogenous of degree 1 in prices:

$$e(\lambda p_1;\lambda p_2;\bar{u})=3\sqrt[3]{\frac{\bar{u}\lambda p_1\lambda^2p_2^2}{4}}=3\lambda\sqrt[3]{\frac{\bar{u}p_1p_2^2}{4}}=\lambda e(p_1;p_2;\bar{u})$$

<u>Problem 3</u> *(Expenditure Minimization, Duality, Constant Elasticity of Substitution)*

Barbara's expenditure minimization problem:

$$\min_{\left\{(x_1;x_2)\in\mathbb{R}_0^{2+}\right\}} \left\{x_1p_1 + x_2p_2 \mid (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} \geq \bar{u}\right\}$$

Since preferences are locally non-satiated, the utility constraint is satisfied with equality,

$$(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} = \bar{u}$$

For $\rho \in (-\infty; 1)$, note that

$$MRS = \frac{x_1^{\rho-1}}{x_2^{\rho-1}} = \frac{x_2^{1-\rho}}{x_1^{1-\rho}}$$

and hence, as $x_1 \to 0$, $MRS \to \infty$ and as $x_2 \to 0$, $MRS \to 0$. It follows that for any values of p_1 and p_2, and any values of x_1 and x_2,

$$\begin{aligned} \frac{p_1}{p_2} &> MRS\,(x_1; 0) \\ \frac{p_1}{p_2} &< MRS\,(0; x_2) \end{aligned}$$

and the problem has no corner solutions. Hence, in the optimum,

$$\begin{aligned} MRS &= \frac{x_2^{1-\rho}}{x_1^{1-\rho}} = \frac{p_1}{p_2} \\ (x_1^\rho + x_2^\rho)^{\frac{1}{\rho}} &= \bar{u} \end{aligned}$$

$$\begin{aligned} x_2 &= \left(\frac{p_1}{p_2}\right)^{\frac{1}{1-\rho}} x_1 \\ \left(x_1^\rho + \left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}} x_1^\rho\right)^{\frac{1}{\rho}} &= \bar{u} \end{aligned}$$

It follows that Barbara's Hicks' demands for burgers and chips satisfy:

$$\begin{aligned} h_1\,(p_1; p_2; \bar{u}) &= \frac{\bar{u}}{\left(1 + \left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} \\ h_2\,(p_1; p_2; \bar{u}) &= \frac{\bar{u}}{\left(1 + \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} \end{aligned}$$

Homogeneity of degree 0 in prices:

$$h_1(\lambda p_1; \lambda p_2; \bar{u}) = \frac{\bar{u}}{\left(1 + \left(\frac{\lambda p_1}{\lambda p_2}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} = \frac{\bar{u}}{\left(1 + \left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} = h_1(p_1; p_2; \bar{u})$$

(analogous for $i = 2$).

Demand for good i is strictly decreasing in p_i:

$$\frac{\partial h_1(p_1; p_2; \bar{u})}{\partial p_1} = -\frac{\bar{u}\frac{1}{1-\rho}\left(\frac{p_1^{2\rho-1}}{p_2^{\rho}}\right)^{\frac{1}{1-\rho}}\left(1 + \left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}-1}}{\left(1 + \left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{2}{\rho}}} < 0$$

(analogous for $i = 2$).

b) Barbara's expenditure function:

$$\begin{aligned}
e(p_1; p_2; \bar{u}) &= \frac{\bar{u}p_1}{\left(1 + \left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} + \frac{\bar{u}p_2}{\left(1 + \left(\frac{p_2}{p_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} \\
&= \frac{\bar{u}p_1 p_2^{\frac{1}{1-\rho}}}{\left((p_2)^{\frac{\rho}{1-\rho}} + (p_1)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} + \frac{\bar{u}p_1^{\frac{1}{1-\rho}} p_2}{\left((p_2)^{\frac{\rho}{1-\rho}} + (p_1)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} \\
&= \frac{\bar{u}p_1 p_2\left(p_2^{\frac{\rho}{1-\rho}} + p_1^{\frac{\rho}{1-\rho}}\right)}{\left((p_2)^{\frac{\rho}{1-\rho}} + (p_1)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} = \frac{\bar{u}p_1 p_2}{\left((p_2)^{\frac{\rho}{1-\rho}} + (p_1)^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}
\end{aligned}$$

c) From the indirect utility function, we have

$$v(p_1; p_2; e(p_1; p_2; \bar{u})) = e(p_1; p_2; \bar{u})\frac{\left(p_1^{\frac{\rho}{1-\rho}} + p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}{p_1 p_2} = \bar{u}$$

$$e(p_1; p_2; \bar{u}) = \frac{\bar{u}p_1 p_2}{\left(p_1^{\frac{\rho}{1-\rho}} + p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}$$

From the Walrasian demands, we obtain

$$
\begin{aligned}
f_1(p_1;p_2;e(p_1;p_2;\bar u)) &= \frac{\bar u p_1 p_2}{\left(p_1^{\frac{\rho}{1-\rho}}+p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}\frac{1}{p_1\left(1+\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}\right)} = \frac{\bar u p_1 p_2}{\left(p_1^{\frac{\rho}{1-\rho}}+p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}\frac{p_2^{\frac{\rho}{1-\rho}}}{p_1\left(p_2^{\frac{\rho}{1-\rho}}+p_1^{\frac{\rho}{1-\rho}}\right)} \\
&= \frac{\bar u p_2^{\frac{1}{1-\rho}}}{\left(p_1^{\frac{\rho}{1-\rho}}+p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} = \frac{\bar u}{\left(\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{1-\rho}}+1\right)^{\frac{1}{\rho}}} = h_1(p_1;p_2;\bar u) \\
f_2(p_1;p_2;e(p_1;p_2;\bar u)) &= \frac{\bar u p_1 p_2}{\left(p_1^{\frac{\rho}{1-\rho}}+p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}\frac{1}{p_2\left(1+\left(\frac{p_2}{p_1}\right)^{\frac{\rho}{1-\rho}}\right)} = \frac{\bar u p_1 p_2}{\left(p_1^{\frac{\rho}{1-\rho}}+p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1-\rho}{\rho}}}\frac{p_1^{\frac{\rho}{1-\rho}}}{p_2\left(p_2^{\frac{\rho}{1-\rho}}+p_1^{\frac{\rho}{1-\rho}}\right)} \\
&= \frac{\bar u p_1^{\frac{1}{1-\rho}}}{\left(p_1^{\frac{\rho}{1-\rho}}+p_2^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} = \frac{\bar u}{\left(\left(\frac{p_2}{p_1}\right)^{\frac{\rho}{1-\rho}}+1\right)^{\frac{1}{\rho}}} = h_2(p_1;p_2;\bar u)
\end{aligned}
$$

Problem 4 *(Duality)*

a) Note that the Hicksian demands satisfy

$$f_l(p;e(p;\bar u)) = \frac{\alpha_l e(p;\bar u)}{p_l} = h_l(p;\bar u)$$

From Shepard's lemma,

$$h_l(p;\bar u) = \frac{\partial e(p;\bar u)}{\partial p_l}$$

Hence,

$$
\begin{aligned}
\frac{\alpha_l e(p;\bar u)}{p_l} &= \frac{\partial e(p;\bar u)}{\partial p_l} \\
\frac{\partial e(p;\bar u)}{\partial p_l}\frac{p_l}{e(p;\bar u)} &= \alpha_l
\end{aligned}
$$

or, e has a constant elasticity in p_l. Hence, the function e has to be a power function, $e(p;\bar u) = c(\bar u)\prod_{l=1}^{L} p_l^{\alpha_l}$, where $c(\bar u)$ is strictly monotone. This implies

$$\frac{\partial e(p;\bar u)}{\partial p_l}\frac{p_l}{e(p;\bar u)} = \frac{\alpha_l c(\bar u)\prod_{\substack{l'=1\\ l'\neq l}}^{L} p_{l'}^{\alpha_{l'}} p_l^{\alpha_l-1}}{c(\bar u)\prod_{l'=1}^{L} p_{l'}^{\alpha_{l'}}} p_l = \alpha_l$$

Hence,

$$h_l(p;\bar{u}) = \alpha_l c(\bar{u}) \prod_{\substack{l'=1 \\ l'\neq l}}^{L} p_{l'}^{\alpha_{l'}} p_l^{\alpha_l - 1}$$

b) Let, e.g., $u(x) = \prod_{l=1}^{L} x_l^{\alpha_l}$, then

$$f_l(p;y) = \frac{\alpha_l y}{p_l}$$

and

$$v(p;y) = \prod_{l=1}^{L} \left(\frac{\alpha_l y}{p_l}\right)^{\alpha_l}$$

and

$$\begin{aligned} v(p;e(p;\bar{u})) &= \prod_{l=1}^{L} \left(\frac{\alpha_l}{p_l}\right)^{\alpha_l} c(\bar{u}) \prod_{l=1}^{L} p_l^{\alpha_l} = \bar{u} \\ c(\bar{u}) &= \frac{\bar{u}}{\prod_{l=1}^{L} \alpha_l^{\alpha_l}} \end{aligned}$$

Problem 5 *(Duality)*

a) From the Roy's identity:

$$\begin{aligned} f_1(p_1;p_2;y) &= -\frac{\partial v(p_1;p_2;y)/\partial p_1}{\partial v(p_1;p_2;y)/\partial y} = \frac{\frac{1}{3}\frac{y}{p_1^{\frac{4}{3}} p_2^{\frac{2}{3}}}}{\frac{1}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}} = \frac{1}{3}\frac{y}{p_1} \\ f_2(p_1;p_2;y) &= -\frac{\partial v(p_1;p_2;y)/\partial p_2}{\partial v(p_1;p_2;y)/\partial y} = \frac{\frac{2}{3}\frac{y}{p_1^{\frac{1}{3}} p_2^{\frac{5}{3}}}}{\frac{1}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}}} = \frac{2}{3}\frac{y}{p_2} \end{aligned}$$

b) To obtain the expenditure function, observe that:

$$\begin{aligned} v(p_1;p_2;e(p_1;p_2;\bar{u})) &= \bar{u} \\ \frac{e(p_1;p_2;\bar{u})}{p_1^{\frac{1}{3}} p_2^{\frac{2}{3}}} &= \bar{u} \\ e(p_1;p_2;\bar{u}) &= \bar{u} p_1^{\frac{1}{3}} p_2^{\frac{2}{3}} \end{aligned}$$

c) From Shepard's lemma:

$$\begin{aligned} h_1(p_1; p_2; \bar{u}) &= \frac{\partial e(p_1; p_2; \bar{u})}{\partial p_1} = \frac{1}{3}\bar{u}\left(\frac{p_2}{p_1}\right)^{\frac{2}{3}} \\ h_2(p_1; p_2; \bar{u}) &= \frac{\partial e(p_1; p_2; \bar{u})}{\partial p_2} = \frac{2}{3}\bar{u}\left(\frac{p_1}{p_2}\right)^{\frac{1}{3}} \end{aligned}$$

d) MC's utility function:

$$u(x) = x_1^{\frac{1}{3}} x_2^{\frac{2}{3}}$$

Problem 6 *(Welfare Evaluation of Price Effects)*

Note that the Hicksian and the Walrasian demand for good 1 (the price of which is subject to change) are given by:

$$\begin{aligned} f_1(p_1; p_2; y) &= \frac{\alpha y}{p_1} \\ h_1(p_1; p_2; y) &= \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \bar{u}\left(\frac{p_2}{p_1}\right)^{1-\alpha} \end{aligned}$$

The indirect utility function at income y_0 is:

$$v(p_1; p_2; y_0) = \frac{\alpha^\alpha (1-\alpha)^{1-\alpha} y_0}{p_1^\alpha p_2^{1-\alpha}}$$

a) The change in your consumer surplus as a result of receiving the gift is given by:

$$CS = \int_{p_1^1}^{p_1^0} \frac{\alpha y_0}{p_1} dp_1 = a y_0 \ln \frac{p_1^0}{p_1^1}$$

b) The minimal sale price you would be willing to accept is equal to the equivalent variation,

$$\begin{aligned} EV &= \int_{p_1^1}^{p_1^0} \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} u^1 \left(\frac{p_2}{p_1}\right)^{1-\alpha} dp_1 \\ &= \int_{p_1^1}^{p_1^0} \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \frac{\alpha^\alpha (1-\alpha)^{1-\alpha} y_0}{(p_1^1)^\alpha p_2^{1-\alpha}} \left(\frac{p_2}{p_1}\right)^{1-\alpha} dp_1 \\ &= \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \frac{\alpha^\alpha (1-\alpha)^{1-\alpha} y_0}{(p_1^1)^\alpha p_2^{1-\alpha}} p_2^{1-\alpha} \int_{p_1^1}^{p_1^0} \left(\frac{1}{p_1}\right)^{1-\alpha} dp_1 \\ &= \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \frac{\alpha^\alpha (1-\alpha)^{1-\alpha} y_0}{(p_1^1)^\alpha p_2^{1-\alpha}} p_2^{1-\alpha} \frac{1}{\alpha} (p_1)^\alpha \Big|_{p_1^1}^{p_1^0} \\ &= \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \frac{\alpha^\alpha (1-\alpha)^{1-\alpha} y_0}{(p_1^1)^\alpha p_2^{1-\alpha}} p_2^{1-\alpha} \frac{1}{\alpha} \left((p_1^0)^\alpha - (p_1^1)^\alpha\right) \end{aligned}$$

c) The maximal price you would be willing to pay for a railcard is the compensated variation:

$$
\begin{aligned}
CV &= \int_{p_1^1}^{p_1^0} \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} u^1 \left(\frac{p_2}{p_1}\right)^{1-\alpha} dp_1 \\
&= \int_{p_1^1}^{p_1^0} \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \frac{\alpha^\alpha (1-\alpha)^{1-\alpha} y_0}{(p_1^0)^\alpha p_2^{1-\alpha}} \left(\frac{p_2}{p_1}\right)^{1-\alpha} dp_1 \\
&= \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \frac{\alpha^\alpha (1-\alpha)^{1-\alpha} y_0}{(p_1^0)^\alpha p_2^{1-\alpha}} p_2^{1-\alpha} \frac{1}{\alpha} \left((p_1^0)^\alpha - (p_1^1)^\alpha\right)
\end{aligned}
$$