Microeconomics – solutions to problem set 5

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Problem 1 *(Production Technology)*

a) A production set that violates irreversibility includes y and $-y$ for some $y \neq 0$. A production set that satisfies it does not.

b) Suppose that $f(\cdot)$ has constant returns to scale, i.e., $f(\alpha v) = \alpha f(v)$ for all $\alpha \geq 0$. If $(y; v) \in Y$, or equivalently, $y \leq f(v)$, then $\alpha y \leq \alpha f(v) = f(\alpha v)$ and hence, $(\alpha y; \alpha v) \in Y$. Suppose now that for every $y \in Y$, $\alpha \geq 0$ implies $\alpha y \in Y$ In particular, for any input vector v, $(v; f(v)) \in Y$ and hence, $(\alpha v; \alpha f(v)) \in Y$. It follows that $\alpha f(v) \leq f(\alpha v)$. Similarly, since $(\alpha v; f(\alpha v)) \in Y$ and $\left(v; \frac{f(\alpha v)}{\alpha}\right) \in Y$, it follows that $f(\alpha v) \leq \alpha f(v)$. Hence, for any v and $\alpha \geq 0$, $f(\alpha v) = \alpha f(v)$

c) Suppose that Y is convex. Then for $(v; f(v))$, $(v'; f(v')) \in Y$, we have for any $\alpha \in (0; 1)$,

$$(\alpha v + (1-\alpha) v'; \alpha f(v) + (1-\alpha) f(v')) \in Y$$

It follows that

$$\alpha f(v) + (1-\alpha) f(v') \leq f(\alpha v + (1-\alpha) v')$$

or that f is concave.

Conversely, assume that $f(\cdot)$ is concave, and hence, for any v, v', $\alpha \in (0; 1)$,

$$\alpha f(v) + (1-\alpha) f(v') \leq f(\alpha v + (1-\alpha) v')$$

Hence, for any $(v; y)$, $(v'; y') \in Y$, $y \leq f(v)$, $y' \leq f(v')$ and, therefore,

$$\alpha y + (1-\alpha) y' \leq \alpha f(v) + (1-\alpha) f(v') \leq f(\alpha v + (1-\alpha) v')$$

implying that $(\alpha v + (1-\alpha) v'; \alpha y + (1-\alpha) y') \in Y$, or that Y is convex.

d) Suppose that $f(v)$ is not convex at some v.

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Then, there is an $\lambda \in (0;1)$ sufficiently close to 1 such that

$$f(\lambda v + (1-\lambda)\cdot 0) > \lambda f(v) + (1-\lambda) f(0) = \lambda f(v)$$

Denoting by $\alpha =: \frac{1}{\lambda} > 1$ and by $v' =: \frac{v}{\lambda}$ we obtain

$$\lambda f(v') > f(\lambda v')$$

in contradiction to increasing returns to scale. The average product is given by $\frac{f(v)}{v}$ and it is increasing iff

$$\frac{f'(v)\,v - f(v)}{v^2} \geq 0 \tag{1}$$

Note that at $v = 0$, the convexity of f implies that $f'(0) \neq \infty$ and, hence, $f'(0)\cdot 0 - f(0) = 0$. Furthermore,

$$(f'(v)\,v - f(v))' = f''(v)\,v + f'(v) - f'(v) = f''(v)\,v > 0$$

Hence, the numerator of (1) is 0 at 0 and increasing. It follows that (1) is always non-negative and hence, $\frac{f(v)}{v}$ is increasing.

Problem 2 *(Profit Maximization, Cost Functions)*

a) The production set satisfies: non-empty, closed, no free lunch, possibility of inaction, free disposal. It does not satisfy: non-increasing and non-decreasing returns to scale, additivity, convexity.

b) Profit maximization:

$$\max_{y,v\geq 0} \{\pi(y;v) = py - wv \mid y \leq f(v)\}$$

In an interior optimum,

$$\begin{aligned} MP &= \frac{w}{p} \\ y &= f(v) \\ \pi(y;v) &\geq 0 \end{aligned}$$

If $v \geq 2$, then $f(v) = 4\sqrt{v-2}$, we obtain

$$v(p;w) = 2 + \frac{4p^2}{w^2} \geq 2$$

The profit is positive if

$$\begin{aligned}\pi(w) &= 4p\sqrt{\frac{4p^2}{w^2}} - 2w - \frac{4p^2}{w} \\ &= \frac{4p^2}{w} - 2w \geq 0 \\ \frac{w}{p} &\leq \sqrt{2}\end{aligned}$$

Hence, the demand for input is given by:

$$v(p;w) = \begin{cases} 2 + \frac{4p^2}{w^2} & \text{if } \frac{w}{p} < \sqrt{2} \\ \left\{0; 2 + \frac{4p^2}{w^2}\right\} & \text{if } \frac{w}{p} = \sqrt{2} \\ 0 & \text{if } \frac{w}{p} > \sqrt{2} \end{cases}$$

The supply of output is:

$$y(p;w) = \begin{cases} \frac{2p}{w} & \text{if } \frac{w}{p} < \sqrt{2} \\ \left\{0; \frac{2p}{w}\right\} & \text{if } \frac{w}{p} = \sqrt{2} \\ 0 & \text{if } \frac{w}{p} > \sqrt{2} \end{cases}$$

c) Cost minimization problem

$$\min_{v \geq 0} \{wv \mid f(v) \geq \bar{y}\}$$

In the optimum, $f(v) = \bar{y}$. Hence,

$$\hat{v}(w) = 2 + \frac{\bar{y}^2}{16}$$

Cost function:

$$c(w;y) = 2w + \frac{y^2}{16}$$

Average cost:

$$\frac{c(w;y)}{y} = \frac{2w}{y} + \frac{y}{16}$$

Marginal cost:

$$c'(w;y) = \frac{y}{8}$$

d) The profit function:

$$\pi(p;w) = \begin{cases} \frac{4p^2}{w} - 2w & \text{if } \frac{w}{p} < \sqrt{2} \\ 0 & \text{if } \frac{w}{p} \geq \sqrt{2} \end{cases}$$

Convexity:

$$\begin{aligned}
& \pi\left(\alpha p+(1-\alpha) p^{\prime} ; \alpha w+(1-\alpha) w^{\prime}\right) \\
= & \frac{4\left(\alpha p+(1-\alpha) p^{\prime}\right)^2}{\alpha w+(1-\alpha) w^{\prime}}-2\left[\alpha w+(1-\alpha) w^{\prime}\right] \\
\leq & \alpha\left[\frac{4\left(\alpha p+(1-\alpha) p^{\prime}\right)^2}{w}-2 w\right]+(1-\alpha)\left[\frac{4\left(\alpha p+(1-\alpha) p^{\prime}\right)^2}{w^{\prime}}-2 w^{\prime}\right]
\end{aligned}$$

$$\begin{aligned}
\leq & \alpha\left[\alpha \frac{4 p^2}{w}+(1-\alpha) \frac{4\left(p^{\prime}\right)^2}{w}-2 w\right]+(1-\alpha)\left[\alpha \frac{4 p^2}{w}+(1-\alpha) \frac{4\left(p^{\prime}\right)^2}{w}-2 w^{\prime}\right] \\
= & \alpha\left[\alpha \frac{4 p^2}{w}+(1-\alpha) \frac{4 p^2}{w}-2 w\right]+(1-\alpha)\left[\alpha \frac{4\left(p^{\prime}\right)^2}{w}+(1-\alpha) \frac{4\left(p^{\prime}\right)^2}{w}-2 w^{\prime}\right] \\
= & \alpha\left[\frac{4 p^2}{w}-2 w\right]+(1-\alpha)\left[\frac{4\left(p^{\prime}\right)^2}{w}-2 w^{\prime}\right] \leq \alpha \pi(p ; w)+(1-\alpha) \pi\left(p^{\prime} ; w^{\prime}\right)
\end{aligned}$$

where the last inequality follows from the fact that if $\frac{w}{p}>\sqrt{2}$

$$\pi(p ; w)=0>\frac{4 p^2}{w}-2 w$$

Hotelling's Lemma:

$$y(p ; w)=\frac{\partial \pi(p ; w)}{\partial p}=\begin{cases}\frac{\partial\left(\frac{4 p^2}{w}-2 w\right)}{\partial p}=\frac{2 p}{w}, & \frac{w}{p}<\sqrt{2} \\ 0, & \frac{w}{p}>\sqrt{2}\end{cases}$$

$$v(p ; w)=-\frac{\partial \pi(p ; w)}{\partial w}=\begin{cases}-\frac{\partial\left(\frac{4 p^2}{w}-2 w\right)}{\partial w}=2+\frac{4 p^2}{w^2}, & \frac{w}{p}<\sqrt{2} \\ 0, & \frac{w}{p}>\sqrt{2}\end{cases}$$

Problem 3 *(Profit maximization with multiple factors of production)*

a) Marginal products:

$$\begin{aligned}
MP_1 &= v_1^{-\frac{3}{4}} v_2^{\frac{1}{4}} \\
MP_2 &= v_1^{\frac{1}{4}} v_2^{-\frac{3}{4}}
\end{aligned}$$

Marginal rate of technical substitution:

$$MRTS=\frac{v_2}{v_1}$$

Decreasing returns to scale, since:

$$f(\lambda v_1; \lambda v_2) = 4\lambda^{\frac{1}{2}} v_1^{\frac{1}{4}} v_2^{\frac{1}{4}} < 4\lambda v_1^{\frac{1}{4}} v_2^{\frac{1}{4}} = \lambda f(v_1; v_2)$$

b) Profit maximization problem:

$$\max_{\{y, v_1, v_2 \geq 0\}} \left\{ py - w_1 v_1 - w_2 v_2 \mid y \leq f(v_1; v_2) = 4 v_1^{\frac{1}{4}} v_2^{\frac{1}{4}} \right\}$$

In the interior optimum,

$$\begin{aligned} MP_1 &= \frac{w_1}{p} \\ MP_2 &= \frac{w_2}{p} \\ y &= f(v_1; v_2) \end{aligned}$$

Since at $v_1 = 0$, $MP_1 = \infty$ and at $v_2 = 0$, $MP_2 = \infty$, there are no corner solutions.

The demand for factors of production is given by:

$$\begin{aligned} v_1(p; w_1; w_2) &= \frac{p^2}{w_1^{\frac{3}{2}} w_2^{\frac{1}{2}}} \\ v_2(p; w_1; w_2) &= \frac{p^2}{w_1^{\frac{1}{2}} w_2^{\frac{3}{2}}} \end{aligned}$$

The supply of output is:

$$y(p; w_1; w_2) = 4 \left(\frac{p^2}{w_1^{\frac{3}{2}} w_2^{\frac{1}{2}}} \frac{p^2}{w_1^{\frac{1}{2}} w_2^{\frac{3}{2}}} \right)^{\frac{1}{4}} = \frac{4p}{w_1^{\frac{1}{2}} w_2^{\frac{1}{2}}}$$

The profit function:

$$\pi(p; w_1; w_2) = py(p; w_1; w_2) - w_1 v_1(p; w_1; w_2) - w_2 v_2(p; w_1; w_2) = \frac{2p^2}{w_1^{\frac{1}{2}} w_2^{\frac{1}{2}}}$$

(Note that the profit is always positive, i.e., there are no solutions for which $y = 0$).

Problem 4 *(Profit maximization with multiple factors of production)*

a) Marginal products:

$$\begin{aligned} MP_1 &= v_1^{-\frac{1}{2}} \\ MP_2 &= 1 \end{aligned}$$

Marginal rate of technical substitution:

$$MRTS = \frac{1}{v_1^{\frac{1}{2}}}$$

b) Profit maximization problem:

$$\max_{\{y,v_1,v_2 \geq 0\}} \{py - w_1 v_1 - w_2 v_2 \mid y \leq f(v_1; v_2) = 2\sqrt{v_1} + v_2\}$$

In the interior optimum,

$$\begin{aligned} MP_1 &= \frac{w_1}{p} \\ MP_2 &= \frac{w_2}{p} \\ y &= f(v_1; v_2) \end{aligned}$$

Since at $v_2 = 0$, $MP_2 = 1$, we have to check for corner solutions.

The demand for factors of productions:

$$\begin{aligned} v_1(p; w_1; w_2) &= \frac{p^2}{w_1^2} \\ v_2(p; w_1; w_2) &= \begin{cases} 0, & \frac{w_2}{p} > 1 \\ [0; \infty), & \frac{w_2}{p} = 1 \\ \infty, & \frac{w_2}{p} < 1 \end{cases} \end{aligned}$$

The supply of output is:

$$y(p; w_1; w_2) = \begin{cases} 2\frac{p}{w_1}, & \frac{w_2}{p} > 1 \\ \left[2\frac{p}{w_1}; \infty\right), & \frac{w_2}{p} = 1 \\ \infty, & \frac{w_2}{p} < 1 \end{cases}$$

c) The profit function:

$$\begin{aligned} \pi(p; w_1; w_2) &= py(p; w_1; w_2) - w_1 v_1(p; w_1; w_2) - w_2 v_2(p; w_1; w_2) \\ &= \begin{cases} 2\frac{p^2}{w_1} & \text{if } \frac{w_2}{p} \geq 1 \\ \infty & \text{if } \frac{w_2}{p} < 1 \end{cases} \end{aligned}$$

Problem 5

a) Profit maximization problem:

$$\max_{\{y,v_1,v_2\geq 0\}} \left\{py - w_1v_1 - w_2v_2 \mid y \leq f(v_1;v_2) = 2\sqrt{v_1+v_2}\right\}$$

Note that

$$MRTS = 1$$

Hence, $v_1 = 0$ if $w_1 > w_2$, $v_2 = 0$ if $w_2 > w_1$. The profit maximization problem thus reduces to:

$$\max_{\{y,v_1,v_2\geq 0\}} \left\{py - \min\{w_1;w_2\}\, v \mid y = 2\sqrt{v}\right\}$$

where $v = v_1 + v_2$ and in the optimum,

$$v = \frac{p^2}{(\min\{w_1;w_2\})^2}$$

Hence, the demand for factors of productions:

$$v_1(p;w_1;w_2) = \begin{cases} 0, & \frac{w_1}{w_2} > 1 \\ \left[0;\frac{p^2}{w_1^2}\right), & \frac{w_1}{w_2} = 1 \\ \frac{p^2}{w_1^2}, & \frac{w_1}{w_2} < 1 \end{cases}$$

$$v_2(p;w_1;w_2) = \begin{cases} \frac{p^2}{w_2^2}, & \frac{w_1}{w_2} > 1 \\ \frac{p^2}{w_2^2} - v_1(p;w_1;w_2), & \frac{w_1}{w_2} = 1 \\ 0, & \frac{w_1}{w_2} < 1 \end{cases}$$

The supply of output is:

$$y(p;w_1;w_2) = \begin{cases} \frac{2p}{w_2}, & \frac{w_1}{w_2} \geq 1 \\ \frac{2p}{w_1}, & \frac{w_1}{w_2} < 1 \end{cases}$$

The profit function:

$$\begin{aligned} \pi(p;w_1;w_2) &= py(p;w_1;w_2) - w_1v_1(p;w_1;w_2) - w_2v_2(p;w_1;w_2) \\ &= \begin{cases} \frac{p^2}{w_2}, & \frac{w_1}{w_2} \geq 1 \\ \frac{2p^2}{w_1}, & \frac{w_1}{w_2} < 1 \end{cases} \end{aligned}$$

b) Profit maximization problem:

$$\max_{\{y,v_1,v_2\geq 0\}} \left\{py - w_1v_1 - w_2v_2 \mid y \leq f(v_1;v_2) = \sqrt{\min\{v_1;v_2\}}\right\}$$

Note that $MRTS = \infty$ if $v_2 > v_1$ and $MRTS = 0$ if $v_1 > v_2$. Hence, in the optimum,

$$v_1 = v_2 =: v$$

The profit maximization problem thus reduces to:

$$\max_{\{y,v_1,v_2 \geq 0\}} \left\{ py - (w_1 + w_2)\, v \mid y = \sqrt{v} \right\}$$

and in the optimum,

$$v = \frac{p^2}{4\,(w_1 + w_2)^2}$$

Hence, the demand for factors of productions:

$$v_1\,(p; w_1; w_2) = v_2\,(p; w_1; w_2) = \frac{p^2}{4\,(w_1 + w_2)^2}$$

The supply of output is:

$$y\,(p; w_1; w_2) = \frac{p}{2\,(w_1 + w_2)}$$

The profit function:

$$\pi\,(p; w_1; w_2) = py\,(p; w_1; w_2) - w_1 v_1\,(p; w_1; w_2) - w_2 v_2\,(p; w_1; w_2) = \frac{p^2}{4\,(w_1 + w_2)}$$

c) Profit maximization problem:

$$\max_{\{y,v_1,v_2 \geq 0\}} \left\{ py - w_1 v_1 - w_2 v_2 \mid y \leq f\,(v_1; v_2) = (v_1^\rho + v_2^\rho)^{\frac{1}{\rho}} \right\}$$

Note that this technology has constant returns to scale. This means that the optimal y is either ∞, or $[0; \infty)$, or 0. In the optimum, $y = f\,(v_1; v_2)$.

$$MRTS = \frac{v_2^{1-\rho}}{v_1^{1-\rho}}$$

and $MRTS = \infty$ if $v_1 = 0$, $MRTS = 0$ if $v_2 = 0$, except for $\rho = 1$, which is the linear case (almost identical to a)). Hence, if output is positive, then both factors of production are used and

$$MRTS = \frac{v_2^{1-\rho}}{v_1^{1-\rho}} = \frac{w_1}{w_2}$$

or

$$v_2 = \left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}} v_1$$

Hence, the optimization problem reduces to:

$$\max_{\{y,v_1,v_2\geq 0\}}\left\{pv_1\left(1+\left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}-\left(w_1+w_2\left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}\right)v_1\right\}$$

and in the optimum,

$$v_1\left(p;w_1;w_2\right)=\begin{cases}0, & p<\frac{\left(w_1+w_2\left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}\right)}{\left(1+\left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}}\\ \left[0;\infty\right), & p=\frac{\left(w_1+w_2\left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}\right)}{\left(1+\left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}}\\ \infty, & p>\frac{\left(w_1+w_2\left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}\right)}{\left(1+\left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}}\end{cases}$$

$$v_2\left(p;w_1;w_2\right)=\begin{cases}0, & p<\frac{\left(w_1+w_2\left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}\right)}{\left(1+\left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}}\\ \left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}v_1, & p=\frac{\left(w_1+w_2\left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}\right)}{\left(1+\left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}}\\ \infty, & p>\frac{\left(w_1+w_2\left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}\right)}{\left(1+\left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}}\end{cases}$$

The supply of output is:

$$y\left(p;w_1;w_2\right)=\begin{cases}0, & p<\frac{\left(w_1+w_2\left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}\right)}{\left(1+\left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}}\\ \left[0;\infty\right), & p=\frac{\left(w_1+w_2\left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}\right)}{\left(1+\left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}}\\ \infty, & p>\frac{\left(w_1+w_2\left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}\right)}{\left(1+\left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}}\end{cases}$$

The profit function:

$$\pi\left(p;w_1;w_2\right) = py\left(p;w_1;w_2\right)-w_1v_1\left(p;w_1;w_2\right)-w_2v_2\left(p;w_1;w_2\right)$$

$$= \begin{cases} 0, & p \leq \frac{\left(w_1 + w_2 \left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}\right)}{\left(1 + \left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} \\ \infty, & p > \frac{\left(w_1 + w_2 \left(\frac{w_2}{w_1}\right)^{\frac{1}{1-\rho}}\right)}{\left(1 + \left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} \end{cases}$$

Problem 6 *(Profit Maximization, Factor Demand, Supply)*

a) Obvious.

b) Use Hotelling's Lemma:

$$\begin{aligned} y_1(p_1; p_2; w_1; w_2) &= \frac{\partial \pi(p_1; p_2; w_1; w_2)}{\partial p_1} \\ &= \frac{\partial \left(\frac{p_1^2}{4(w_1 - p_2)} + \frac{p_1^2}{4(w_2 - p_2)}\right)}{\partial p_1} \\ &= p_1 \left(\frac{1}{2(w_1 - p_2)} + \frac{1}{2(w_2 - p_2)}\right) \end{aligned}$$

$$\begin{aligned} y_2(p_1; p_2; w_1; w_2) &= \frac{\partial \pi(p_1; p_2; w_1; w_2)}{\partial p_2} \\ &= \frac{\partial \left(\frac{p_1^2}{4(w_1 - p_2)} + \frac{p_1^2}{4(w_2 - p_2)}\right)}{\partial p_2} \\ &= p_1^2 \left(\frac{1}{16(w_1 - p_2)^2} + \frac{1}{16(w_2 - p_2)^2}\right) \end{aligned}$$

$$\begin{aligned} v_1(p_1; p_2; w_1; w_2) &= -\frac{\partial \pi(p_1; p_2; w_1; w_2)}{\partial w_1} \\ &= -\frac{\partial \left(\frac{p_1^2}{4(w_1 - p_2)} + \frac{p_1^2}{4(w_2 - p_2)}\right)}{\partial w_2} \\ &= \frac{p_1^2}{16(w_1 - p_2)^2} \end{aligned}$$

$$\begin{aligned} v_2(p_1; p_2; w_1; w_2) &= -\frac{\partial \pi(p_1; p_2; w_1; w_2)}{\partial w_2} \\ &= -\frac{\partial \left(\frac{p_1^2}{4(w_1 - p_2)} + \frac{p_1^2}{4(w_2 - p_2)}\right)}{\partial w_2} \\ &= \frac{p_1^2}{16(w_2 - p_2)^2} \end{aligned}$$

c) Production function:

$$f\left(v_1; v_2\right) = \left(2\left(\sqrt{v_1} + \sqrt{v_2}\right); v_1 + v_2\right)$$