# Microeconomics – solutions to problem set 6

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## Problem 1

**a)** Short-run cost minimization problem:

$$\min_{v_2 \geq 0} \left\{ w_1 \bar{v}_1 + w_2 v_2 \mid 2\sqrt{v_1 v_2} \geq \bar{y} \right\}$$

Demand for $v_2$:

$$v_2 (w_1; w_2; \bar{y}) = \frac{\bar{y}^2}{4\bar{v}_1} = \frac{\bar{y}^2}{16}$$

Short-run cost function:

$$\tilde{c} (w_1; w_2; y) = 4w_1 + \frac{w_2 y^2}{16}$$

**b)** Average cost:

$$\frac{\tilde{c} (w_1; w_2; y)}{y} = 4\frac{w_1}{y} + \frac{w_2 y}{16}$$

Marginal cost:

$$\tilde{c}' (w_1; w_2; y) = \frac{w_2 y}{8}$$

Short-run supply of $y$:

$$\begin{aligned} p &= \tilde{c}' (w_1; w_2; y) \\ y (p; w_1; w_2) &= \frac{8p}{w_2} \end{aligned}$$

**c)** Long-run cost-minimization problem:

$$\min_{\left\{(v_1; v_2) \in \mathbb{R}_0^{2+}\right\}} \left\{ w_1 v_1 + w_2 v_2 \mid 2\sqrt{v_1 v_2} \geq \bar{y} \right\}$$

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In the optimum,

$$\begin{aligned} MRTS &= \frac{v_2}{v_1} = \frac{w_1}{w_2} \\ 2\sqrt{v_1 v_2} &= \bar{y} \end{aligned}$$

$$\begin{aligned} v_1 &= \frac{\bar{y}^2}{4v_2} \\ v_2 &= \frac{w_1}{w_2} v_1 = \frac{w_1}{w_2} \frac{\bar{y}^2}{4v_2} \end{aligned}$$

Demand for inputs:

$$\begin{aligned} v_2 (w_1; w_2; \bar{y}) &= \sqrt{\frac{w_1}{w_2}} \frac{\bar{y}}{2} \\ v_1 (w_1; w_2; \bar{y}) &= \sqrt{\frac{w_2}{w_1}} \frac{\bar{y}}{2} \end{aligned}$$

Cost function $c(w_1; w_2; y)$:

$$c(w_1; w_2; y) = \sqrt{w_1 w_2} y$$

Optimal supply of $y$:

$$y(w_1; w_2; p) = \begin{cases} +\infty & \text{if } p > \sqrt{w_1 w_2} \\ [0; +\infty) & \text{if } p = \sqrt{w_1 w_2} \\ 0 & \text{if } p < \sqrt{w_1 w_2} \end{cases}$$

**d)** The short-run and the long-run costs are equal if:

$$\begin{aligned} c(w_1; w_2; y) &= \tilde{c}(w_1; w_2; y) \\ \sqrt{w_1 w_2} y &= 4w_1 + \frac{w_2 y^2}{16} \\ (8\sqrt{w_1} - \sqrt{w_2} y)^2 &= 0 \end{aligned}$$

$$\hat{y} = 8\sqrt{\frac{w_1}{w_2}}$$

Obviously,

$$c(w_1; w_2; y) - \tilde{c}(w_1; w_2; y) = (8\sqrt{w_1} - \sqrt{w_2} y)^2 \geq 0$$

since in the long-run both factors of production can be freely adjusted.

Problem 2 *(Cost Minimization, Conditional Factor Demand, Profit Maximization)*

**a)** Cost minimization problem:

$$\min_{(v_1...v_M)\in R_0^{M+}} \left\{ \sum_{i=1}^{M} w_i v_i \mid \sum_{i=1}^{M} \sqrt{v_i} \geq \bar{y} \right\}$$

$$MRTS_{ij} = \sqrt{\frac{v_j}{v_i}}$$

Since for every $i$, at $v_i = 0$, $MRTS_{ij} = \infty$, the problem has no corner solutions.

**b)** In the optimum,

$$\begin{aligned} MRTS_{ij} &= \sqrt{\frac{v_j}{v_i}} = \frac{w_i}{w_j} \\ \sum_{i=1}^{M} \sqrt{v_i} &= \bar{y} \end{aligned}$$

Hence, for any $i \in \{2...M\}$,

$$\begin{aligned} v_i &= \frac{v_1 w_1^2}{w_i^2} \\ \sqrt{v_1} w_1 \sum_{i=1}^{M} \frac{1}{w_i} &= \bar{y} \end{aligned}$$

Conditional factor demand function:

$$v_i (w_1...w_M) = \frac{\bar{y}^2}{w_i^2 \left( \sum_{j=1}^{M} \frac{1}{w_j} \right)^2}$$

Homogeneity of degree 0:

$$v_i (\lambda w_1...\lambda w_M) = \frac{\bar{y}^2}{\lambda^2 w_i^2 \left( \sum_{j=1}^{M} \frac{1}{\lambda w_j} \right)^2} = \frac{\bar{y}^2}{w_i^2 \left( \sum_{j=1}^{M} \frac{1}{w_j} \right)^2} = v_i (w_1...w_M)$$

**c)** The cost function of the firm:

$$c (w_1...w_M; y) = \sum_{i=1}^{M} \frac{y^2}{w_i \left( \sum_{j=1}^{M} \frac{1}{w_j} \right)^2} = \frac{y^2}{\left( \sum_{i=1}^{M} \frac{1}{w_i} \right)}$$

Marginal costs:

$$c' (w_1...w_M; y) = \frac{2y}{\left( \sum_{i=1}^{M} \frac{1}{w_i} \right)}$$

Average costs:

$$\frac{c(w_1...w_M;y)}{y} = \frac{y}{\left(\sum_{i=1}^{M}\frac{1}{w_i}\right)}$$

Homogeneity of degree 1 in $w$:

$$c(\lambda w_1...\lambda w_M;y) = \frac{y^2}{\left(\sum_{i=1}^{M}\frac{1}{\lambda w_i}\right)} = \frac{\lambda y^2}{\left(\sum_{i=1}^{M}\frac{1}{w_i}\right)} = \lambda c(w_1...w_M;y)$$

Concavity in $w$:

$$\begin{aligned} c(\alpha w + (1-\alpha) w';y) &= \frac{y^2}{\left(\sum_{i=1}^{M}\frac{1}{\alpha w_i + (1-\alpha) w_i'}\right)} \geq \frac{y^2}{\sum_{i=1}^{M}\left(\frac{\alpha}{w_i} + \frac{(1-\alpha)}{w_i'}\right)} \\ &= y^2 \frac{1}{\alpha\sum_{i=1}^{M}\frac{1}{w_i} + (1-\alpha)\sum_{i=1}^{M}\frac{1}{w_i'}} \geq \end{aligned}$$

$$\begin{aligned} &\geq y^2\left[\alpha\frac{1}{\sum_{i=1}^{M}\frac{1}{w_i}} + (1-\alpha)\frac{1}{\sum_{i=1}^{M}\frac{1}{w_i'}}\right] \\ &= \alpha c(w;y) + (1-\alpha) c(w';y) \end{aligned}$$

where the first inequality follows from the fact that $\frac{1}{w_i}$ is a convex function, whereas the second inequality follows from the fact that $\sum_{i=1}^{M}\frac{1}{w_i}$ is a sum of convex functions and thus, a convex function itself.

Non-decreasing and convex in $y$: obvious.

**d)** Derive the conditional factor demand from the cost function: Shepard's Lemma:

$$v_i(w_1...w_M;y) = \frac{\partial c(w_1...w_M;y)}{\partial w_i} = \partial\frac{y^2}{\left(\sum_{i=1}^{M}\frac{1}{w_i}\right)}/\partial w_i = \frac{y^2}{w_i^2\left(\sum_{i=1}^{M}\frac{1}{w_i}\right)^2}$$

**e)** The supply function of the firm:

$$\begin{aligned} p &= c'(w_1...w_M;y) \\ p &= \frac{2y}{\left(\sum_{i=1}^{M}\frac{1}{w_i}\right)} \\ y(p;w_1...w_M) &= \frac{p}{2}\left(\sum_{i=1}^{M}\frac{1}{w_i}\right) \end{aligned}$$

The profit function:

$$\pi\left(p; w_1 ... w_M\right) = py\left(p; w_1 ... w_M\right) - c\left(w_1 ... w_M; y\left(p; w_1 ... w_M\right)\right) = \frac{p^2}{4}\left(\sum_{i=1}^{M} \frac{1}{w_i}\right)$$

<u>Problem 3</u> *(Cost Minimization)*

**a)** Conditional factor demand:

$$\min_{\left\{(v_1; v_2) \in \mathbb{R}_0^{2+}\right\}} \left\{w_1 v_1 + w_2 v_2 \mid v_1 + v_2 \leq \bar{y}\right\}$$

Since $MRTS = 1$, in the optimum

$$v_2 = \bar{y} - v_1$$

and

$$v_1\left(w_1; w_1; \bar{y}\right) = \begin{cases} \bar{y} & \text{if } \frac{w_1}{w_2} < 1 \\ [0; \bar{y}] & \text{if } \frac{w_1}{w_2} = 1 \\ 0 & \text{if } \frac{w_1}{w_2} > 1 \end{cases}$$

$$v_2\left(w_1; w_1; \bar{y}\right) = \begin{cases} 0 & \text{if } \frac{w_1}{w_2} < 1 \\ \bar{y} - v_1 & \text{if } \frac{w_1}{w_2} = 1 \\ \bar{y} & \text{if } \frac{w_1}{w_2} > 1 \end{cases}$$

Cost function:

$$\begin{aligned} c\left(w_1; w_2; y\right) &= w_1 v_1\left(w_1; w_1; \bar{y}\right) + w_2 v_2\left(w_1; w_1; \bar{y}\right) \\ &= \begin{cases} w_1 \bar{y} & \text{if } \frac{w_1}{w_2} \leq 1 \\ w_2 \bar{y} & \text{if } \frac{w_1}{w_2} > 1 \end{cases} \end{aligned}$$

**b)** Conditional factor demand:

$$\min_{\left\{(v_1; v_2) \in \mathbb{R}_0^{2+}\right\}} \left\{w_1 v_1 + w_2 v_2 \mid \min\left\{v_1; v_2\right\} \leq \bar{y}\right\}$$

Since $MRTS = +\infty$ if $v_2 > v_1$ and $MRTS = 0$ if $v_2 < v_1$, in the optimum

$$v_1\left(w_1; w_1; \bar{y}\right) = v_2\left(w_1; w_1; \bar{y}\right) = \bar{y}$$

Cost function:

$$c\left(w_1; w_2; y\right) = \bar{y}\left(w_1 + w_2\right)$$

**c)** Conditional factor demand:

$$\min_{\left\{(v_1;v_2)\in\mathbb{R}_0^{2+}\right\}} \left\{w_1 v_1 + w_2 v_2 \mid \left(v_1^{\rho} + v_2^{\rho}\right)^{\frac{1}{\rho}} \leq \bar{y}\right\}$$

In the optimum,

$$\begin{aligned} MRTS &= \frac{v_2^{1-\rho}}{v_1^{1-\rho}} = \frac{w_1}{w_2} \\ \left(v_1^{\rho} + v_2^{\rho}\right)^{\frac{1}{\rho}} &= \bar{y} \end{aligned}$$

$$v_2 = \left(\frac{w_1}{w_2}\right)^{\frac{1}{1-\rho}} v_1$$

and substituting into the second condition,

$$\begin{aligned} v_1\left(w_1; w_2; \bar{y}\right) &= \frac{\bar{y}}{\left(1 + \left(\frac{w_1}{w_2}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} \\ v_2\left(w_1; w_2; \bar{y}\right) &= \frac{\bar{y}}{\left(1 + \left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} \end{aligned}$$

Cost function:

$$\begin{aligned} c\left(w_1; w_2; y\right) &= w_1 v_1\left(w_1; w_1; \bar{y}\right) + w_2 v_2\left(w_1; w_1; \bar{y}\right) \\ &= \frac{\bar{y} w_1}{\left(1 + \left(\frac{w_1}{w_2}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} + \frac{\bar{y} w_2}{\left(1 + \left(\frac{w_2}{w_1}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{1}{\rho}}} \end{aligned}$$

## Problem 4 *(Efficiency)*

**a)** Increasing returns to scale for $v \in [0; 12]$, decreasing for $v > 12$.

Is the production set convex? No.

**b)** Profit maximization problem:

$$\max_{\{v\geq 0\}} \left\{p \frac{1}{16} v^2 - v\right\}$$

Demand for labor

$$v\left(p\right) = \begin{cases} 0 & \text{if } p < \frac{4}{3} \\ \{0; 12\} & \text{if } p = \frac{4}{3} \\ 12 & \text{if } p > \frac{4}{3} \end{cases}$$

Supply of cookies:

$$y(p) = \begin{cases} 0 & \text{if } p < \frac{4}{3} \\ \{0; 9\} & \text{if } p = \frac{4}{3} \\ 9 & \text{if } p > \frac{4}{3} \end{cases}$$

Profit function:

$$\pi(p) = \begin{cases} 0 & \text{if } p \leq \frac{4}{3} \\ 9p - 12 & \text{if } p > \frac{4}{3} \end{cases}$$

At $p \leq \frac{4}{3}$, producing no cookies is optimal.

**c)** Luna's utility maximization problem:

$$\max_{\{f \in [0;12]; k \geq 0\}} \{fk \mid pk \leq (12 - f) + \pi\}$$

Luna's labor supply is $l = 12 - f$ and we can rewrite her optimization problem as:

$$\max_{\{l \in [0;12]; k \geq 0\}} \{(12 - l)\, k \mid pk \leq l + \pi\}$$

In the optimum,

$$\begin{aligned} MRS &= \frac{12 - l}{k} = p \\ pk &= l + \pi \end{aligned}$$

Supply of labor:

$$l(p; \pi) = \begin{cases} 6 - \frac{\pi}{2}, & \pi \leq 12 \\ 0, & \pi > 12 \end{cases}$$

Demand for cookies:

$$k(p; \pi) = \begin{cases} \frac{12+\pi}{2p}, & \pi \leq 12 \\ \frac{\pi}{p}, & \pi > 12 \end{cases}$$

**d)** Is there a price $p$ at which Luna's demand for cookies equals the supply?

$$k(p; \pi(p)) = y(p)$$

Suppose first that $p \geq \frac{4}{3}$ and, thus, $y(p) = 9$, $\pi(p) = 9p - 12$. If $\pi(p) \leq 12$, substituting in the equation above, we obtain $\frac{9}{2} = 9$, a contradiction. If $\pi(p) > 12$, then $l(p; \pi) = 0$, hence, the firm faces 0 labor supply and cannot produce any cookies. If $p < \frac{4}{3}$, then $\pi(p) = y(p) = 0$. Hence, $k(p; \pi) = \frac{6}{p} \neq y(p)$. So there is no price $p$ at which the market for cookies is in equilibrium. This is due to the non-convexity of the production set and thus, the non-convexity of the firm's supply function.

**e)** A production plan, for which Luna's marginal rate of substitution equals the marginal rate of transformation of the cookie machine:

$$
\begin{aligned}
MRS &= \frac{\frac{l^2}{16}}{12-l} = MP = \frac{1}{8}l \\
l &= 8, k = 4
\end{aligned}
$$

Does this production plan maximize Luna's utility subject to the production set of the economy? Yes:

$$
\begin{aligned}
&\max_{l\in[0;12]} \frac{l^2}{16}(12-l) \\
l &= 8, k = 4
\end{aligned}
$$

The production plan is efficient, but not profit-maximizing.

<u>Problem 5</u> *(Production, Aggregation)*

Note that for any firm $j$, $c_j'(y_j) = \alpha + 2\beta_j y_j$. Cost-minimization implies

$$
\begin{aligned}
&\min_{(y_j\geq 0)_{j=1}^J} \left\{ \sum_{j=1}^J c_j(y_j) \mid \sum_{j=1}^J y_j = y \right\} \\
= &\min_{(y_j\geq 0)_{j=1}^J} \left\{ \sum_{j=1}^{J-1} c_j(y_j) + c_J\left(y - \sum_{j=1}^{J-1} y_j\right) \right\}
\end{aligned}
$$

**a)** If $\beta_j > 0$ for all $j$, then the total cost function is convex and hence, in the optimumum, for every $J$,

$$
\begin{aligned}
c_j'\left(y_j^*\right) &= c_J'\left(y_J^*\right) \\
\sum_{j=1}^J y_j^* &= y
\end{aligned}
$$

We obtain:

$$y_j^* = \frac{\beta_J}{\beta_j} y_J^*$$

and

$$y_j^* = \frac{y}{\beta_j \sum_{j=1}^J \frac{1}{\beta_j}}, j \in J$$

**b) and c)** If some $\beta_j < 0$, the cost function is no longer convex. Since marginal cost is decreasing in output, whenever $\beta_j < 0$, it makes sense to concentrate production on those firms $j$ with the smalles negative $\beta_j$. Hence, the set of optimal production plans is:

$$\left\{ (y_1^* ... y_J^*) \mid \sum_{j \in \arg\min\{\beta_j\}} y_j^* = y, y_j^* = 0, j \notin \arg\min \{\beta_j\} \right\}$$